On the determinantal structure of conditional overlaps for the complex Ginibre ensemble

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Abstract

We continue the study of joint statistics of eigenvectors and eigenvalues initiated in the seminal papers of Chalker and Mehlig. The principal object of our investigation is the expectation of the matrix of overlaps between the left and the right eigenvectors for the complex $N \times N$ Ginibre ensemble, conditional on an arbitrary number $k = 1, 2, \ldots$ of complex eigenvalues. These objects provide the simplest generalisation of the expectations of the diagonal overlap ($k = 1$) and the off-diagonal overlap ($k = 2$) considered originally by Chalker and Mehlig. They also appear naturally in the problem of joint evolution of eigenvectors and eigenvalues for Brownian motions with values in complex matrices studied by the Krakow school.

We find that these expectations possess a determinantal structure, where the relevant kernels can be expressed in terms of certain orthogonal polynomials in the complex plane. Moreover, the kernels admit a rather tractable expression for all $N \geq 2$. This result enables a fairly straightforward calculation of the conditional expectation of the overlap matrix in the local bulk and edge scaling limits as well as the proof of the exact algebraic decay and asymptotic factorisation of these expectations in the bulk.

1 Introduction and Motivation.

Let $\mathrm{Gin}(N, \mathbb{C})$ be an ensemble of $N \times N$ matrices with independent complex Gaussian entries (complex Ginibre ensemble): if $M \sim \mathrm{Gin}(N, \mathbb{C})$ is a complex Ginibre matrix, then

$$\mathbb{E}(M_{ij}) = 0 = \mathbb{E}(\bar{M}_{ij}),\ 1 \leq i, j \leq N, \tag{1.1}$$

$$\mathbb{E}(M_{ij}M_{kl}) = 0 = \mathbb{E}(\bar{M}_{ij}\bar{M}_{kl}),\ 1 \leq i, j, k, l \leq N, \tag{1.2}$$

$$\mathbb{E}(M_{ij}\bar{M}_{kl}) = \delta_{ik}\delta_{jl}\ 1 \leq i, j, k, l \leq N, \tag{1.3}$$

where '$^-$' stands for complex conjugation. Let $\mathbf{\Lambda}^{(N)} = (\Lambda_1, \Lambda_2, \ldots, \Lambda_N)$ be the set of complex eigenvalues of M. This ensemble was introduced in 1965 in [19] along with its real and quaternionic counterparts. It was immediately realised in this pioneering paper that the marginal distribution of eigenvalues for $\mathrm{Gin}(N, \mathbb{C})$ can be computed and is a natural generalisation of the corresponding answer for the Gaussian Unitary Ensemble (GUE), cf. [27], to the case of a complex spectrum:

$$p_N\left(\mathbf{\Lambda}^{(N)} \in d\boldsymbol{\lambda}^{(N)}\right) d\boldsymbol{\lambda}^{(N)} = \frac{1}{Z_N}|\Delta^{(N)}(\boldsymbol{\lambda}^{(N)})|^2 e^{-\sum_{j=1}^N |\lambda_j|^2} d\boldsymbol{\lambda}^{(N)}. \tag{1.4}$$

Here p_N is the density for the distribution of the eigenvalues with respect to Lebesgue measure $d\boldsymbol{\lambda}^{(N)} = \prod_{k=1}^N d\lambda_k\bar{\lambda}_k$ on $\mathbb{C}^N$, $\Delta^{(N)}(\boldsymbol{\lambda}^{(N)}) = \prod_{i>j}^N(\lambda_i - \lambda_j)$ is the Vandermonde determinant, and $Z_N = \pi^N \prod_{j=1}^N j!$ is the normalisation constant. As one can see, p_N can be interpreted as the coordinate distribution function of the classical log-gas in two dimensions, which is parallel to the view of the ensemble of GUE eigenvalues as the one-dimensional log-gas. It is perhaps this analogy that biased the research into $\mathrm{Gin}(N,\mathbb{C})$ towards the study of its spectral properties, see e.g. [27, 16, 22] for reviews of many significant results concerning the spectrum of $\mathrm{Gin}(N,\mathbb{C})$. Yet, there is an important and nowadays widely appreciated difference between the complex Ginibre ensemble and GUE: even though the marginal distribution of eigenvalues for $\mathrm{Gin}(N,\mathbb{C})$ can be computed analytically, the statistics of eigenvectors does not decouple from the statistics of eigenvalues. Despite this fundamental difference, it was not until the late nineties that Chalker and Mehlig initiated the quantitative study of the joint statistics of eigenvectors and eigenvalues for $\mathrm{Gin}(N,\mathbb{C})$. They were motivated by questions of spectral stability for non-Hermitian random Hamiltonians describing certain quantum or stochastic complex systems, a problem which can be traced back to [25]. For further motivations from Physics see [8] and references therein.

Stability questions can be hard to justify within a static setting of the complex Ginibre ensemble, but become very natural if one considers some kind of dynamics (random or deterministic) on the space of complex matrices. Our own interest in the statistics of eigenvectors for $\mathrm{Gin}(N,\mathbb{C})$ was inspired by the study of the stochastic dynamics of eigenvectors and eigenvalues of complex matrices initiated by Z. Burda and M. Nowak, their collaborators and students ('The Krakow school'), see e.g. [21, 8] and references therein. So, in order to motivate the main subject of this paper, let us follow [7] and [20] and consider the Brownian motion M_t with values in $N \times N$ complex matrices started from zero. In other words, $(M_t)_{t\geq 0}$ is a Gaussian process with continuous paths, independent increments and the covariance

$$\mathbb{E}\left(\mathrm{Tr}\left(A^\dagger M_t^\dagger\right)\mathrm{Tr}\left(M_s B\right)\right) = \mathrm{Tr}\left(A^\dagger B\right), \tag{1.5}$$

where A, B are arbitrary $N \times N$ complex matrices. The fixed time $t > 0$ marginal law for the process $(M_t)_{t\geq 0}$ coincides up to rescaling with the complex Ginibre ensemble.

Let $(\Lambda_{t\alpha}, \boldsymbol{L}_{t\alpha}, \boldsymbol{R}_{t\alpha})_{t\geq 0, 1\leq\alpha\leq N}$ be the induced processes, describing the evolution of eigenvalues of M_t and the bi-orthogonal set of corresponding left and right eigenvectors,

$$\boldsymbol{L}_{t\alpha}^\dagger M_t = \Lambda_{t\alpha}\boldsymbol{L}_{t\alpha}^\dagger,\ 1\leq\alpha\leq N, \tag{1.6}$$

$$M_t\boldsymbol{R}_{t\alpha} = \Lambda_{t\alpha}\boldsymbol{R}_{t\alpha},\ 1\leq\alpha\leq N, \tag{1.7}$$

$$\langle \boldsymbol{L}_{t\alpha}, \boldsymbol{R}_{t\beta}\rangle = \delta_{\alpha,\beta},\ 1\leq\alpha,\beta\leq N, \tag{1.8}$$

where ' $\dagger$ ' denotes Hermitean conjugation and $\langle\cdot,\cdot\rangle$ stands for the Hermitean inner product on $\mathbb{C}^N$. As shown in [7] and [20], the process $(\Lambda_{t\alpha})_{t\geq 0, 1\leq\alpha\leq N}$ is a complex martingale such that

$$d\Lambda_{t\alpha}d\bar{\Lambda}_{t\beta} = O_{t\alpha\beta}dt, \tag{1.9}$$

where

$$O_{t\alpha\beta} = \langle \boldsymbol{L}_{t\alpha}, \boldsymbol{L}_{t\beta}\rangle\langle \boldsymbol{R}_{t\alpha}, \boldsymbol{R}_{t\beta}\rangle,\ 1\leq\alpha,\beta\leq N \tag{1.10}$$

is the matrix of the overlaps between the left and the right eigenvectors of M_t. (It is worth noticing that paper [20] derives the full set of stochastic differential equations for the joint evolution of eigenvalues and eigenvectors of M_t for any matrix size N.) Notice that for complex matrices, the matrix of overlaps is a non-trivial random variable as the left and the right eigenvectors are not orthogonal,

$$\langle \boldsymbol{L}_\alpha, \boldsymbol{L}_\beta\rangle \neq 0, \langle \boldsymbol{R}_\alpha, \boldsymbol{R}_\beta\rangle \neq 0,\ 1\leq\alpha<\beta\leq N.$$

As a result, the evolution of eigenvalues for complex matrices is very different from the case of normal matrices with complex spectrum, despite both models having the same marginal distribution of eigenvalues for zero initial conditions. See Appendix A for more details on the dynamics of eigenvalues for normal matrices.

To study the evolution of eigenvalues corresponding to (1.9), it is natural to study conditional expectations

$$\mathbb{E}_N(d\Lambda_{t\alpha}d\bar{\Lambda}_{t\alpha} \mid \Lambda_{t\alpha} = \lambda_\alpha) = \mathbb{E}(O_{t\alpha\alpha} \mid \Lambda_{t\alpha} = \lambda_\alpha)dt,\ 1 \leq \alpha \leq N,$$

and

$$\mathbb{E}_N(d\Lambda_{t\alpha}d\bar{\Lambda}_{t\beta} \mid \Lambda_{t\alpha} = \lambda_\alpha, \Lambda_{t\beta} = \lambda_\beta) = \mathbb{E}(O_{t\alpha\beta} \mid \Lambda_{t\alpha} = \lambda_\alpha, \Lambda_{t\beta} = \lambda_\beta)dt,\ 1 \leq \alpha \neq \beta \leq N,$$

where $\mathbb{E}_N(\cdot)$ denotes expectation with respect to $\mathrm{Gin}(N, \mathbb{C})$. These are the conditional expectations of the diagonal and the non-diagonal overlaps originally studied in [11, 26]. Furthermore, if we wish to understand the influence of a fixed set of eigenvalues on the evolution of a single eigenvalue or a pair of eigenvalues, it is reasonable to consider general conditional expectations

$$\mathbb{E}_N(O_{t\alpha_1\alpha_2} \mid \Lambda_{t\ \alpha_p} = \lambda_{\alpha_p}, p = 1, 2, \ldots k),\ 1 \leq \alpha_p \leq N, k = 1, 2, \ldots N.$$

These are the principal objects studied in the present paper. An additional motivation for our study comes from the mathematical structure of the answers: we find that conditional expectations of overlaps are expressed in terms of determinants of matrices built out of a kernel of some integrable operator. While this structure is a well-known feature of point processes associated with the statistics of eigenvalues of random matrices, we were unaware of determinantal answers for the statistics of eigenvectors prior to starting our work.

Our work continues the mathematical study of the statistical properties of eigenvectors of non-Hermitian matrices, which has become an active research area during the past few years. This renewed effort has already yielded a number of significant generalisations of the original results by Chalker and Mehlig: In a breakthrough paper [7], Bourgade and Dubach prove that the law of the diagonal overlap conditional on the corresponding eigenvalue is given in the bulk scaling limit by the inverse Gamma-distribution with parameter 2. This is a significant generalisation of the results of Chalker and Mehlig who managed to calculate this distribution for the matrix size $N = 2$ only. The statement follows from a beautiful novel representation of the diagonal overlap conditioned on all eigenvalues as a product of independent random variables. The authors also obtain new results for the variance of the off-diagonal overlaps and the two-point function of diagonal overlaps, establishing in particular the algebraic decay of the latter as a function of the distance between the corresponding eigenvalues. In a parallel development [17], Fyodorov obtains the full conditional law of the diagonal overlap both for the real and complex Ginibre ensembles. Fyodorov's answer is valid for $N < \infty$, which allows him to derive the scaling limits for the distribution of the diagonal overlap both in the bulk and near the edge of the spectrum as $N \to \infty$. Of course, the answers of [17] are consistent with that of [7]. The calculations in [17] are based on a novel representation of the distribution of the diagonal overlap in terms of ratios of determinants and employs the calculus of anti-commuting variables, cf. [18] for an alternative analytical approach. In [33], Walters and Starr extend the answers of [11, 26] for the conditional expectation of the diagonal overlap at $N < \infty$ to any conditioned value of the corresponding eigenvalue. This allows the authors to calculate the edge scaling limit for the conditional expectation of the diagonal overlap. It is worth stressing that our own calculations are based on the same analysis of recursion relations for the determinants of certain 3-diagonal moment matrices as in [33]. We complement it by an exact correspondence between diagonal and off-diagonal overlaps, which allows us to avoid difficulties associated with the analysis of the 5-diagonal moment matrices. In [13], Crawford and Rosenthal study high order moments of the overlap matrix (1.10). They prove the existence of the bulk scaling limit for the moments and discover a beautiful factorization relation, valid on a macroscopic scale, expressing the moments of an arbitrary order in terms of a linear combination of products of moments of order two, the structure of which deserves further investigation. In an investigation having a slightly different flavour, the authors of [24] and [30] prove the delocalisation property of eigenvectors for ensembles of complex random matrices, which do not necessarily possess unitary invariance. The delocalisation property means that the weight of the coefficients is not concentrated in any particular region of the index space. In [5], the authors study the statistics of angles between the eigenvectors for invariant non-Gaussian ensembles. Finally, we must mention the work of the Krakow School, which is at least partially responsible for the current renaissance of research into the joint statistics of eigenvectors and eigenvalues for random non-Hermitian matrices. Among its recent contributions most relevant to the present work is the derivation of the system of stochastic evolution equations for eigenvalues and eigenvectors, cf. [20], which allowed its authors to express the rate of change of eigenvalue correlation functions in terms of conditional expectations of overlaps. These are precisely the objects studied in the present paper; in [28], its authors present evidence for the microscopic universality of moments of overlaps by exploiting a perturbative expansion in N^{-1} for the calculation of moments for non-Gaussian ensembles of complex matrices. In contrast, on a macroscopic scale it is shown in [4], combining free probability and the methods of generalised Green's functions [21], that the eigenvector correlators depend on the radial spectral cumulative distribution in a concise way, and thus is non-universal.

The rest of the paper is organised as follows. Section 2 presents our main results concerning conditional expectations of overlaps: the determinantal representation for $N < \infty$, the bulk and the edge scaling limits, exact algebraic asymptotic in the bulk for well separated eigenvalues. Section 3 contains the proofs in the following subsections : 3.1, 3.2 the derivation of the determinantal representation for the conditional expectations of diagonal and off-diagonal overlaps in terms of bi-orthogonal polynomials in the complex plane; 3.3 a heuristic calculation of the correlation kernels, which shows how the result of rather complicated calculations of the following sections can be easily guessed using the assumption of the extended translational invariance; 3.4 a rigorous evaluation of correlation kernels for $N < \infty$ in terms of the exponential polynomials; 3.5 - 3.8 the calculation of various scaling limits as $N \to \infty$. Appendix A contains the derivation of Dyson-like stochastic evolution equations for the normal matrix model.

The methods used in the proofs are rather classical: the determinantal structure is a consequence of Dyson's theorem reviewed in [27] and the product structure of the overlap expectations conditioned on all eigenvalues; the computation of the correlation kernel for the diagonal overlaps reduces to the inversion of the tri-diagonal moment matrix using the recursions already encountered in [11, 26] and [33]; the calculation of the kernel for the off-diagonals overlaps uses a relation between diagonal and off-diagonal overlaps established in Lemma 1 and determinantal identities, see [23] for a review.

2 Statement and Discussion of Results.

As already explained in the introduction, we will be interested in the joint statistics of the overlaps and eigenvalues of $M \sim \mathrm{Gin}(N, \mathbb{C})$. Namely, we will study the following conditional expectations:

$$\mathbb{E}_N(O_{\alpha\alpha} \mid \Lambda_m = \lambda_m, m \in I), \ I \subset \{1, 2, \dots, N\}, \alpha \in I \tag{2.1}$$

$$\mathbb{E}_N(O_{\alpha\beta} \mid \Lambda_m = \lambda_m, m \in J), \ J \subset \{1, 2, \dots, N\}, \alpha, \beta \in J. \tag{2.2}$$

In other words , we consider the expectation of the overlaps with respect to $\mathrm{Gin}(N, \mathbb{C})$ measure conditioned on a set of eigenvalues. To be more concrete, if $M \sim \mathrm{Gin}(N, \mathbb{C})$ is parametrised using Schur coordinates, we compute the expected overlaps with respect to the product measure whose factors are the Haar measure for the unitary conjugation, a Gaussian measure for the upper triangular degrees of freedom, and the eigenvalue measure obtained by conditioning (1.4) on a set of eigenvalues. Due to the permutation symmetry of $\mathrm{Gin}(N, \mathbb{C})$ measure, it is sufficient to consider the following expectations:

$$\mathbb{E}_N(O_{11} \mid \Lambda_1 = \lambda_1, \Lambda_2 = \lambda_2, \ldots \Lambda_k = \lambda_k), \ k = 1, 2, \ldots, N, \tag{2.3}$$
$$\mathbb{E}_N(O_{12} \mid \Lambda_1 = \lambda_1, \Lambda_2 = \lambda_2, \ldots \Lambda_k = \lambda_k), \ k = 2, \ldots, N. \tag{2.4}$$

Closely associated with these expectations are the following weighted multi-point intensities of the eigenvalues:

$$D_{11}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) := \mathbb{E}_N(O_{11} \mid \Lambda_1 = \lambda_1, \ldots, \Lambda_k = \lambda_k)\rho^{(N,k)}(\boldsymbol{\lambda}^{(k)}), \quad k = 1, 2, \ldots, N, \tag{2.5}$$

and

$$D_{12}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) := \mathbb{E}_N(O_{12} \mid \Lambda_1 = \lambda_1, \ldots, \Lambda_k = \lambda_k)\rho^{(N,k)}(\boldsymbol{\lambda}^{(k)}), \quad k = 2, \ldots, N, \tag{2.6}$$

where $\boldsymbol{\lambda}^{(k)} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\rho^{(N,k)}$ is the k-point correlation function (Lebesgue density for factorial moments) for $\mathrm{Gin}(N, \mathbb{C})$ eigenvalues. Recall that

$$\rho^{(N,k)}(\boldsymbol{\lambda}^{(k)}) := \frac{N!}{(N-k)!} \int_{\mathbb{C}^{N-k}} \prod_{m=k+1}^{N} d\lambda_m d\bar{\lambda}_m p_N(\boldsymbol{\lambda}^{(N)}) = \det_{1 \le i,j \le N} \left(K_{ev}^{(N)}(\lambda_i, \lambda_j) \right), \tag{2.7}$$

where

$$K_{ev}^{(N)}(x, y) = \frac{1}{\pi} e^{-|x|^2} \sum_{m=0}^{N-1} \frac{(\bar{x}y)^m}{m!} \tag{2.8}$$

is the kernel of the determinantal point process corresponding to the distribution of $\mathrm{Gin}(N, \mathbb{C})$ eigenvalues, see [19] and [27] for the derivation of (2.7) and (2.8). [1]

For the sake of brevity, we will refer to the expectations (2.5) and (2.6) as conditional overlaps. Notice that

$$D_{11}^{(N,1)}(\lambda) = \mathbb{E}_N \left(\sum_{\alpha=1}^{N} O_{\alpha\alpha} \delta(\Lambda_\alpha - \lambda) \right), \tag{2.9}$$
$$D_{12}^{(N,2)}(\lambda, \mu) = \mathbb{E}_N \left(\sum_{\alpha \neq \beta = 1}^{N} O_{\alpha\beta} \delta(\Lambda_\alpha - \lambda) \delta(\Lambda_\beta - \mu) \right), \tag{2.10}$$

coincide with the expectations of diagonal and off-diagonal elements of the overlap matrix studied by Chalker and Mehlig: compare $D_{11}^{(N,1)}(\lambda)$ and $D_{12}^{(N,2)}(\lambda, \mu)$ with equations (10) and (11) of [26] evaluated at $\sigma = 1$.

Our starting point is the fundamental result of [11, 26] for the overlaps conditioned on *all* eigenvalues:

$$D_{11}^{(N,N)}(\boldsymbol{\lambda}^{(N)}) = N! \prod_{k=2}^{N} \left(1 + \frac{1}{|\lambda_1 - \lambda_k|^2} \right) p_N(\boldsymbol{\lambda}^{(N)}), \tag{2.11}$$
$$D_{12}^{(N,N)}(\boldsymbol{\lambda}^{(N)}) = -\frac{N!}{|\lambda_1 - \lambda_2|^2} \prod_{k=3}^{N} \left(1 + \frac{1}{(\lambda_1 - \lambda_k)(\bar{\lambda}_2 - \bar{\lambda}_k)} \right) p_N(\boldsymbol{\lambda}^{(N)}), \tag{2.12}$$

[1] The kernel (2.8) can be re-written in a more symmetric form $\frac{1}{\pi} e^{-\frac{1}{2}(|x|^2 + |y|^2)} \sum_{m=0}^{N-1} \frac{(\bar{x}y)^m}{m!}$ by the conjugation $K_{ev}(x, y) \to e^{\frac{1}{2}x^2} K_{ev}(x, y) e^{-\frac{1}{2}y^2}$, which does not change the correlation functions.

see equations (43) and (46) of [26]. Thus the task of integrating over the unitary and upper triangular coordinates has already been accomplished by Chalker and Mehlig and we can concentrate on computing the expectation of $D_{11}^{(N,N)}$, $D_{12}^{(N,N)}$ with respect to the conditional eigenvalue measure. The study of conditional expectations of overlaps is further simplified due to a simple relation between D_{11} and D_{12}: let $\hat{T}$ be the following transposition acting on functions on $\mathbb{C}^{2k}$, $k \geq 2$:

$$\hat{T} f(\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \ldots) = f(\lambda_1, \bar{\lambda}_2, \lambda_2, \bar{\lambda}_1, \ldots). \tag{2.13}$$

We have the following

Lemma 1. *(Exact relation between diagonal and off-diagonal overlaps for $N < \infty$.) For any $2 \leq k \leq N < \infty$, the functions $D_{11}^{(N,k)}$ and $D_{12}^{(N,k)}$ are entire functions on $\mathbb{C}^{2k}$. Moreover,*

$$D_{12}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{e^{-|\lambda_1-\lambda_2|^2}}{1-|\lambda_1-\lambda_2|^2} \hat{T} D_{11}^{(N,k)}(\boldsymbol{\lambda}^{(k)}). \tag{2.14}$$

To state the main result of the paper we need to introduce some notations. Let

$$e_p(x) = \sum_{k=0}^{p} \frac{x^k}{k!},\ p = 0, 1, 2, \ldots \tag{2.15}$$

be the exponential polynomial of order p considered as functions on $\mathbb{C}$. Let

$$f_p(x) = (p+1)e_p(x) - x e_{p-1}(x),\ p = 0, 1, \ldots, \tag{2.16}$$

where we define $e_{-1}(x) \equiv 0$. The polynomials f_p are closely related to the bi-orthogonal polynomials in the complex plane associated with conditional overlaps, see Section 3.4 for details. Finally, let $\mathfrak{F}_n : \mathbb{C}^3 \to \mathbb{C}$ be the following polynomial in three variables:

$$\begin{aligned} \mathfrak{F}_n(x,y,z) &= e_n(xy)\cdot e_n(xz) - e_n(xyz)\cdot e_n(x)\cdot(1 - x(1-y)(1-z)) \\ &+ \frac{(1-y)(1-z)}{n!}\cdot\frac{(xyz)^{n+1}e_n(x) - x^{n+1}e_n(xyz)}{1-yz},\ n = 0, 1, \ldots \end{aligned} \tag{2.17}$$

The following is the main result of the paper:

Theorem 1. *(Determinantal structure of conditional overlaps) For any $1 \leq k \leq N < \infty$,*

$$D_{11}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) = \frac{f_{N-1}(|\lambda_1|^2)}{\pi} e^{-|\lambda_1|^2} \det_{2\leq i,j\leq k}\left(K_{11}^{(N-1)}\left(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1\right)\right), \tag{2.18}$$

where the kernel

$$K_{11}^{(N)}(x, \bar{x}, y, \bar{y} \mid \lambda, \bar{\lambda}) = \omega(x, \bar{x} \mid \lambda, \bar{\lambda})\kappa^{(N)}(\bar{x}, y \mid \lambda, \bar{\lambda}), \tag{2.19}$$

is a function on $\mathbb{C}^6$, which is built out of the weight

$$\omega(x, y \mid \lambda, \mu) = \frac{1}{\pi}(1 + (x-\lambda)(y-\mu))e^{-xy}, \tag{2.20}$$

a function on $\mathbb{C}^4$, and the reduced kernel

$$\kappa^{(N)}(\bar{x}, y \mid \lambda, \bar{\lambda}) = \frac{\left((N+1)\,\mathfrak{F}_{N+1}\left(\lambda\bar{\lambda}, \frac{\bar{x}}{\bar{\lambda}}, \frac{y}{\lambda}\right) - \lambda\bar{\lambda}\mathfrak{F}_N\left(\lambda\bar{\lambda}, \frac{\bar{x}}{\bar{\lambda}}, \frac{y}{\lambda}\right)\right)}{\left(\bar{x}-\bar{\lambda}\right)^2 (y-\lambda)^2 f_N\left(\lambda\bar{\lambda}\right)}. \tag{2.21}$$

Furthermore, for $k \geq 2$,

$$
\begin{aligned}
D_{12}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) &= -\frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{\pi^2} f_{N-1}(\lambda_1\bar{\lambda}_2)\kappa^{(N-1)}(\bar{\lambda}_1,\lambda_2 \mid \lambda_1,\bar{\lambda}_2) \\
&\quad \times \det_{3\leq i,j\leq k}\left(K_{12}^{(N-1)}\left(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j \mid \lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2\right)\right), \qquad (2.22)
\end{aligned}
$$

where

$$
\begin{aligned}
K_{12}^{(N)}(x,\bar{x},y,\bar{y} \mid u,\bar{u},v,\bar{v}) &= \frac{\omega(x,\bar{x} \mid u,\bar{v})}{\kappa^{(N)}(\bar{u},v \mid u,\bar{v})} \qquad (2.23) \\
&\quad \times \det\begin{pmatrix} \kappa^{(N)}(\bar{u},v \mid u,\bar{v}) & \kappa^{(N)}(\bar{u},y \mid u,\bar{v}) \\ \kappa^{(N)}(\bar{x},v \mid u,\bar{v}) & \kappa^{(N)}(\bar{x},y \mid u,\bar{v}) \end{pmatrix}.
\end{aligned}
$$

Remark. Everywhere in the paper we use the convention that the determinant of an empty matrix is equal to 1.

The finite-N answer stated above enables an easy study of the large-N limits of conditional overlaps. It is well known that the global spectral density of complex eigenvalues approaches the circular law, $\lim_{N\to\infty}\rho^{(N,1)}(\sqrt{N}z) = \frac{1}{\pi}\Theta(1-|z|)$, where Θ is the Heaviside step function, cf. [22]. Therefore, we will consider two such local, microscopic limits: the local bulk scaling limit,

$$
D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{N\to\infty}\frac{1}{N}D_{11}^{(N,k)}(\boldsymbol{\lambda}^{(k)}), \qquad (2.24)
$$

$$
D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{N\to\infty}\frac{1}{N}D_{12}^{(N,k)}(\boldsymbol{\lambda}^{(k)}), \qquad (2.25)
$$

i.e. we fix $\lambda_1,\dots,\lambda_k$ and take the large-N limit, which places us in the vicinity of the origin[2], and the local edge scaling limit,

$$
D_{11}^{(edge,\ k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{N\to\infty}\frac{1}{\sqrt{N}}D_{11}^{(N,k)}(e^{i\theta}(\sqrt{N}+\boldsymbol{\lambda}^{(k)})), \qquad (2.26)
$$

$$
D_{12}^{(edge,\ k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{N\to\infty}\frac{1}{\sqrt{N}}D_{12}^{(N,k)}(e^{i\theta}(\sqrt{N}+\boldsymbol{\lambda}^{(k)})), \qquad (2.27)
$$

i.e. we shift to the vicinity of the spectral edge at $|z| = \sqrt{N}$, fix $\lambda_1,\dots,\lambda_k$ and then take the $N\to\infty$ limit. The overall rescaling of conditional overlaps used for the bulk and edge limits by the factors of N^{-1} and $N^{-1/2}$ correspondingly is justified in the introduction. Notice also that our notations for the edge scaling limit reflect the independence of the final answer on the point at the edge of the spectrum around which we expand.

Corollary 1. *(Local bulk scaling limit of conditional overlaps)*

$$
D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = \frac{1}{\pi}\det_{2\leq i,j\leq k}\left(K_{11}^{(bulk)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j \mid \lambda_1,\bar{\lambda}_1)\right), \qquad (2.28)
$$

where $K_{11}^{(bulk)} : \mathbb{C}^6 \to \mathbb{C}$ is the limiting kernel:

$$
K_{11}^{(bulk)}(u,\bar{u},v,\bar{v} \mid \lambda,\bar{\lambda}) = \omega^{(bulk)}(u,\bar{u} \mid \lambda,\bar{\lambda})\kappa^{(bulk)}(\bar{u},v \mid \lambda,\bar{\lambda}), \qquad (2.29)
$$

where

$$
\omega^{(bulk)}(u,\bar{u} \mid \lambda,\bar{\lambda}) = \frac{1}{\pi}(1+(u-\lambda)(\bar{u}-\bar{\lambda}))e^{-(u-\lambda)(\bar{u}-\bar{\lambda})} \qquad (2.30)
$$

[2]To access the general bulk we would have to scale $z = re^{i\theta}\sqrt{N}+\lambda$, with $r<1$ and fixed λ.

is the weight and

$$\kappa^{(bulk)}(\bar{u}, v \mid \lambda, \bar{\lambda}) = \frac{d}{dz}\left(\frac{e^z - 1}{z}\right)\Bigg|_{z=(\bar{u}-\bar{\lambda})(v-\lambda)}, \tag{2.31}$$

is the reduced kernel. Moreover,

$$D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{1}{\pi^2}\kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) \det_{3\le i,j\le k}\left(K_{12}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2)\right), \tag{2.32}$$

where

$$\begin{aligned} K_{12}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2) &= \frac{\omega^{(bulk)}(\lambda_i, \bar{\lambda}_i \mid \lambda_1, \bar{\lambda}_2)}{\kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2)} \\ &\times \det\begin{pmatrix} \kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) & \kappa^{(bulk)}(\bar{\lambda}_1, \lambda_j \mid \lambda_1, \bar{\lambda}_2) \\ \kappa^{(bulk)}(\bar{\lambda}_i, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) & \kappa^{(bulk)}(\bar{\lambda}_i, \lambda_j \mid \lambda_1, \bar{\lambda}_2) \end{pmatrix}. \end{aligned} \tag{2.33}$$

As expected, conditional overlaps in the bulk are translationally invariant, meaning that $D_{11}^{(bulk,\ k)}$ and $D_{12}^{(bulk,\ k)}$ are invariant with respect to a simultaneous shift of the arguments,

$$\lambda_i \to \lambda_i + \mu, \bar{\lambda}_i \to \bar{\lambda}_i + \bar{\mu},\ 1 \le i \le k,\ \mu \in \mathbb{C}.$$

Less trivially, the overlaps in the bulk are invariant with respect to the above transformation for arbitrary complex numbers μ and $\bar{\mu}$, which are not necessarily conjugate to each other. This extended translational invariance is responsible for the success of the short heuristic derivation of Corollary 1 given in Section 3.3.

Notice that the reduced kernel (2.31) on the real line coincides with the density of *eigenvalues* for a truncated unitary ensemble in the regime of weak non-unitarity found by Sommers and Zyczkowski, see Eqn. (21) of [34] at $L = 1$. At the moment we do not understand any deep reason for such a coincidence.

Finally, let us verify that the statement of Corollary 1 agrees with Chalker and Mehlig's answer for $D_{12}^{(bulk,2)}(\lambda_1, \lambda_2)$ obtained in [11, 26]. Specialising (2.32) to the particular case $k = 2$ and denoting

$$\lambda_{ij} = \lambda_i - \lambda_j,\ \bar{\lambda}_{ij} = \bar{\lambda}_i - \bar{\lambda}_j,\ 1 \le i, j \le N, \tag{2.34}$$

we find that

$$\begin{aligned} D_{12}^{(bulk,\ 2)}(\lambda_1, \lambda_2) &= -\frac{1}{\pi^2}\kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) \\ &= -\frac{1}{\pi^2}\frac{d}{dz}\left(\frac{e^z - 1}{z}\right)|_{z=-|\lambda_{12}|^2} = -\frac{1}{\pi^2}\frac{1}{|\lambda_{12}|^4}\left(1 - \left(1 + |\lambda_{12}|^2\right)e^{-|\lambda_{12}|^2}\right), \end{aligned}$$

which corresponds to Eqn. (9) of [11] for fluctuations at the origin ($z_+ = 0$ in [11]). We conjecture the corresponding local bulk kernels (2.29) and (2.33) to be universal, see also [28].

The finite-N results stated in Theorem 1 are also well suited for studying the statistics of overlaps at the edge. For $a \in \mathbb{C}$, let

$$F(a) = \frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-x^2/2}dx \equiv \frac{1}{2}\mathrm{erfc}\left(\frac{a}{\sqrt{2}}\right), \tag{2.35}$$

where erfc is the complementary error function, analytically continued to the complex plane. For any $a, b, c, d, f \in \mathbb{C}$, let

$$\begin{aligned} H(a, b, c, d, f) &= -\frac{\sqrt{2\pi}}{\left(1 - \sqrt{2\pi}ae^{\frac{a^2}{2}}F(a)\right)} \\ &\times \frac{d}{dx}\left[e^{\frac{(a+x)^2}{2}}\left(e^{-f}F(b+x)F(c+x) - F(d+x)F(a+x) + fF(d)F(a+x)\right)\right]\Bigg|_{x=0}. \end{aligned} \tag{2.36}$$

Corollary 2. *(Local edge scaling limit of conditional overlaps)*

$$\begin{aligned} D_{11}^{(edge,\ k)}(\boldsymbol{\lambda}^{(k)}) &= \frac{1}{\sqrt{2\pi^3}}\left(e^{-\frac{1}{2}(\lambda_1+\bar{\lambda}_1)^2} - \sqrt{2\pi}(\lambda_1+\bar{\lambda}_1)F(\lambda_1+\bar{\lambda}_1)\right) \\ &\times \det_{2\le i,j\le k}\left(K_{11}^{(edge)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j \mid \lambda_1,\bar{\lambda}_1)\right), \end{aligned} \tag{2.37}$$

where $K_{11}^{(edge)}:\mathbb{C}^6\to\mathbb{C}$ *is the limiting kernel:*

$$K_{11}^{(edge)}(x,\bar{x},y,\bar{y}\mid\lambda,\bar{\lambda}) = \omega^{(edge)}(x,\bar{x}\mid\lambda,\bar{\lambda})\kappa^{(edge)}(\bar{x},y\mid\lambda,\bar{\lambda}), \tag{2.38}$$

where

$$\omega^{(edge)}(x,\bar{x}\mid\lambda,\bar{\lambda}) = \frac{1}{\pi}(1+(x-\lambda)(\bar{x}-\bar{\lambda}))e^{-x\bar{x}} \tag{2.39}$$

is the weight and

$$\kappa^{(edge)}(\bar{x},y\mid\lambda,\bar{\lambda}) = e^{\bar{x}y}\frac{H\left(\lambda+\bar{\lambda},\lambda+\bar{x},y+\bar{\lambda},y+\bar{x},(\lambda-y)(\bar{\lambda}-\bar{x})\right)}{(\lambda-y)^2(\bar{\lambda}-\bar{x})^2} \tag{2.40}$$

is the reduced kernel. Moreover,

$$\begin{aligned} D_{12}^{(edge,\ k)}(\boldsymbol{\lambda}^{(k)}) &= -\frac{1}{\sqrt{2\pi^5}}\left(1-\sqrt{2\pi}(\lambda_1+\bar{\lambda}_2)e^{\frac{1}{2}(\lambda_1+\bar{\lambda}_2)^2}F(\lambda_1+\bar{\lambda}_2)\right) \\ &\times\frac{e^{-|\lambda_1-\lambda_2|^2-\frac{1}{2}(\lambda_1+\bar{\lambda}_2)^2}}{\lambda_{12}^2\bar{\lambda}_{12}^2}H(\lambda_1+\bar{\lambda}_2,\lambda_1+\bar{\lambda}_1,\lambda_2+\bar{\lambda}_2,\lambda_2+\bar{\lambda}_1,-\lambda_{12}\bar{\lambda}_{12}) \\ &\times\det_{3\le i,j\le k}\left(K_{12}^{(edge)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j\mid\lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2)\right), \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} K_{12}^{(edge)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j\mid\lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2) &= \frac{\omega^{(edge)}(\lambda_i,\bar{\lambda}_i\mid\lambda_1,\bar{\lambda}_2)}{\kappa^{(edge)}(\bar{\lambda}_1,\lambda_2\mid\lambda_1,\bar{\lambda}_2)} \\ &\times\ \det\begin{pmatrix}\kappa^{(edge)}(\bar{\lambda}_1,\lambda_2\mid\lambda_1,\bar{\lambda}_2) & \kappa^{(edge)}(\bar{\lambda}_1,\lambda_j\mid\lambda_1,\bar{\lambda}_2)\\ \kappa^{(edge)}(\bar{\lambda}_i,\lambda_2\mid\lambda_1,\bar{\lambda}_2) & \kappa^{(edge)}(\bar{\lambda}_i,\lambda_j\mid\lambda_1,\bar{\lambda}_2)\end{pmatrix}. \end{aligned} \tag{2.42}$$

As expected, the translational invariance is lost at the edge. However, it is easy to check that $D_{11}^{(edge,\ k)}$ and $D_{12}^{(edge,\ k)}$ are invariant with respect to a global shift along the edge of the spectrum,

$$\lambda_m\to\lambda_m+i\mu,\bar{\lambda}_m\to\bar{\lambda}_m-i\mu,\ 1\le m\le k,\ \mu\in\mathbb{R}.$$

This symmetry is just an infinitesimal version of the global $U(1)$-symmetry of the complex Ginibre ensemble, which survives in the large-N limit.

It follows from the statement of Corollary 2, that for $k=1$,

$$D_{11}^{(edge,\ 1)}(\lambda_1) = \frac{1}{\sqrt{2\pi^3}}\left(e^{-\frac{1}{2}(\lambda_1+\bar{\lambda}_1)^2}-\sqrt{2\pi}(\lambda_1+\bar{\lambda}_1)F(\lambda_1+\bar{\lambda}_1)\right), \tag{2.43}$$

which coincides with the answer for the edge scaling limit of the diagonal overlap obtained in [33, Corollary 4.3]. For $k=2$, we find that

$$\begin{aligned} &D_{12}^{(edge,\ 2)}(\lambda_1,\lambda_2) = \\ &\frac{1}{\pi^2}\frac{e^{-|\lambda_1-\lambda_2|^2-\frac{1}{2}(\lambda_1+\bar{\lambda}_2)^2}}{\lambda_{12}^2\bar{\lambda}_{12}^2}\frac{d}{dx}\Bigg[e^{\frac{(\lambda_1+\bar{\lambda}_2+x)^2}{2}}\Big(e^{\lambda_{12}\bar{\lambda}_{12}}F(\lambda_1+\bar{\lambda}_1+x)F(\lambda_2+\bar{\lambda}_2+x) \\ &-F(\lambda_2+\bar{\lambda}_1+x)F(\lambda_1+\bar{\lambda}_2+x)-\lambda_{12}\bar{\lambda}_{12}F(\lambda_2+\bar{\lambda}_1)F(\lambda_1+\bar{\lambda}_2+x)\Big)\Bigg]\Bigg|_{x=0}, \end{aligned} \tag{2.44}$$

which is apparently a new expression for the off-diagonal overlap at the edge. Again we conjecture the local edge kernels (2.38) and (2.42) to be universal.

As it is easy to check, both the bulk and the edge scaling limits of $D_{11}^{(N,k)}$ and $D_{12}^{(N,k)}$ given in Corollaries 1 and 2 are related via the statement of Lemma 1. This reflects the fact that the large-N limit preserves the analytic properties of conditional overlaps. In particular both the bulk and the edge scaling limits of $D_{11}^{(N,k)}$ and $D_{12}^{(N,k)}$ are entire functions of $\boldsymbol{\lambda}^{(k)}$ and $\bar{\boldsymbol{\lambda}}^{(k)}$.

There is also a different kind of relation between the scaling limits of overlaps: as we have already reviewed, the typical magnitude of the overlap in the bulk is $O(N)$, near the edge - $O(\sqrt{N})$. This is consistent with the fact that the prefactor in (2.37) diverges as we move back into the bulk: if $Re(\lambda) = Re(\bar{\lambda}) = R$,

$$\lim_{R\to-\infty}\left(e^{-(\lambda+\bar{\lambda})^2} - \sqrt{2\pi}(\lambda+\bar{\lambda})F(\lambda+\bar{\lambda})\right) = -\sqrt{2\pi}\lim_{R\to-\infty}(\lambda+\bar{\lambda}) = +\infty.$$

Therefore, there is no *a priori* reason for any relation between conditional overlaps in the bulk and at the edge. However, simple analysis of the answers presented in Corollaries 1 and 2 reveals the following relations:

Corollary 3.

$$\lim_{R\to-\infty}\frac{D_{11}^{(edge,\ k)}(R\mathbf{1}^{(k)}+\boldsymbol{\lambda}^{(k)})}{D_{11}^{(edge,\ 1)}(R+\lambda_1)} = \frac{D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)})}{D_{11}^{(bulk,\ 1)}(\lambda_1)},\ k=1,2,\dots \tag{2.45}$$

$$\lim_{R\to-\infty}\frac{D_{12}^{(edge,\ k)}(R\mathbf{1}^{(k)}+\boldsymbol{\lambda}^{(k)})}{D_{12}^{(edge,\ 2)}(R+\lambda_1,R+\lambda_2)} = \frac{D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)})}{D_{12}^{(bulk,\ 2)}(\lambda_1,\lambda_2)},\ k=2,3,\dots, \tag{2.46}$$

where $\mathbf{1}^{(k)} = (1,1,\dots,1)\in\mathbb{R}^k$.

Notice that a similar relation for the *eigenvalues* is known, see [15], but it is perhaps less surprising, as there is no rescaling involved in the calculation of eigenvalue intensities in the bulk and at the edge.

Conditional overlaps provide a natural measure of dependence between eigenvectors and eigenvalues. Recall, that for $\mathrm{Gin}(N,\mathbb{C})$ the eigenvalue correlations decay exponentially with the square distance between the eigenvalues on a large scale of separation, see e.g. [27]. In contrast, the decay of correlations between eigenvalues and conditional overlaps is algebraic.

Corollary 4. *(Exact algebraic asymptotic for conditional overlaps.) Consider conditional overlaps* D_k^{11} *and* D_k^{12} *in the bulk scaling limit. Suppose the eigenvalues* $\lambda_1,\lambda_2,\dots,\lambda_k$ *are uniformly separated, i.e. there exists* $L>0$*:*

$$|\lambda_{ij}|\geq L,\ 1\leq i<j\leq k.$$

Then, for large values of L*,*

$$D_{11}^{(bulk,k)}(\boldsymbol{\lambda}^{(k)}) = \left(\frac{1}{\pi}\right)^k\prod_{m=2}^{k}\left(1-\frac{1}{|\lambda_{m1}|^4}\right)+O(e^{-L^2}), \tag{2.47}$$

$$D_{12}^{(bulk,k)}(\boldsymbol{\lambda}^{(k)}) = -\left(\frac{1}{\pi}\right)^k\frac{1}{|\lambda_{12}|^4}\prod_{m=3}^{k}\left(1-\frac{1}{\lambda_{m1}^2\bar{\lambda}_{m2}^2}\right)+O(e^{-L^2}). \tag{2.48}$$

Notice the that Corollary 4 implies an asymptotic factorisation of conditional overlaps. Namely, it establishes the existence of functions $P(\cdot\mid\lambda_1)$ and $Q(\cdot\mid\lambda_1,\lambda_2)$ on $\mathbb{C}$ such that

$$\pi^k D_{11}^{(bulk,k)}(\boldsymbol{\lambda}^{(k)}) = \prod_{m=2}^{k}P(\lambda_m\mid\lambda_1)+O(e^{-L^2}),$$

$$\pi^k|\lambda_{12}|^4 D_{12}^{(bulk,k)}(\boldsymbol{\lambda}^{(k)}) = -\prod_{m=3}^{k}Q(\lambda_m\mid\lambda_1,\lambda_2)+O(e^{-L^2}).$$

This statement is a consequence of a relation between conditional overlaps in the bulk and correlation functions for eigenvalues, which might be of independent interest:

$$D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = (-1)^{k-1}\prod_{m=2}^{k}\frac{1+|\lambda_{m1}|^2}{|\lambda_{m1}|^4}\left(1-|\lambda_{m1}|^2-\lambda_{m1}\frac{\partial}{\partial\lambda_m}\right)\rho^{(bulk,k)}(\boldsymbol{\lambda}^{(k)}), \quad (2.49)$$

$$D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = \frac{(-1)^{k-1}}{|\lambda_{12}|^4}\left(1-\lambda_{21}\frac{\partial}{\partial\lambda_2}\right)\prod_{m=3}^{k}\frac{1+\lambda_{m1}\bar{\lambda}_{m2}}{\lambda_{m1}^2\bar{\lambda}_{m2}^2}\left(1-\lambda_{m1}\bar{\lambda}_{m2}-\lambda_{m1}\frac{\partial}{\partial\lambda_m}\right) \times\rho^{(bulk,k)}(\boldsymbol{\lambda}^{(k)}). \quad (2.50)$$

Notice that the differential operators entering the product in the right hand side of (2.49) and (2.50) commute, so there is no ambiguity in the above formulae due to the ordering, see Section 3.8 for the derivation. We conjecture that the algebraic decay and the factorisation property for the conditional overlaps stated in Corollary 4 remains true in the global bulk scaling limit as well.

3 Proofs

3.1 General set-up for the proof of Theorem 1. The determinantal structure.

Recall expressions (2.11) and (2.12) for the overlaps conditioned on N eigenvalues. Averaging over all the eigenvalues but $\lambda_1,\ldots,\lambda_k$, we get

$$D_{11}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) = \frac{1}{Z_N}\frac{N!}{(N-k)!}\int_{\mathbb{C}^{N-k}}\prod_{i=k+1}^{N}d\lambda_i d\bar{\lambda}_i|\Delta^{(N)}(\lambda_1,\lambda_2,\ldots,\lambda_N)|^2 e^{-\sum_{j=1}^N|\lambda_j|^2} \times\prod_{k=2}^{N}\left(1+\frac{1}{|\lambda_1-\lambda_k|^2}\right), \quad (3.1)$$

$$D_{12}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) = \frac{1}{Z_N}\frac{N!}{(N-k)!}\int_{\mathbb{C}^{N-k}}\prod_{i=k+1}^{N}d\lambda_i d\bar{\lambda}_i|\Delta^{(N)}(\lambda_1,\lambda_2,\ldots,\lambda_N)|^2 e^{-\sum_{j=1}^N|\lambda_j|^2} \times\frac{1}{|\lambda_1-\lambda_2|^2}\prod_{k=3}^{N}\left(1+\frac{1}{(\lambda_1-\lambda_k)\left(\bar{\lambda}_2-\bar{\lambda}_k\right)}\right), \quad (3.2)$$

where $Z_N=\pi^N\prod_{j=1}^N j!$ is the normalisation constant. Therefore,

$$D_{11}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) = \frac{e^{-|\lambda_1|^2}}{Z_N}\frac{N!}{(N-k)!}\int_{\mathbb{C}^{N-k}}\prod_{i=k+1}^{N}d\lambda_i d\bar{\lambda}_i|\Delta^{(N-1)}(\lambda_2,\ldots,\lambda_N)|^2 \times\prod_{m=2}^{N}\pi\omega(\lambda_m,\bar{\lambda}_m\mid\lambda_1,\bar{\lambda}_1), \quad (3.3)$$

$$D_{12}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{Z_N}\frac{N!}{(N-k)!}\int_{\mathbb{C}^{N-k}}\prod_{i=k+1}^{N}d\lambda_i d\bar{\lambda}_i\Delta^{(N-1)}(\lambda_2,\lambda_3,\ldots,\lambda_N) \times\Delta^{(N-1)}(\bar{\lambda}_1,\bar{\lambda}_3,\ldots,\bar{\lambda}_N)\prod_{m=3}^{N}\pi\omega(\lambda_m,\bar{\lambda}_m\mid\lambda_1,\bar{\lambda}_2), \quad (3.4)$$

where the integration measure is defined in both cases by the following function on $\mathbb{C}^3$:

$$\omega(z,\bar{z}|u,v)=\frac{1}{\pi}(1+(z-u)(\bar{z}-v))e^{-z\bar{z}},\ \ z,u,v\in\mathbb{C}. \quad (3.5)$$

In order to determine $D_{12}^{(N,k)}$ using Lemma 1, proved in Section 3.2 below, we need to calculate $D_{11}^{(N,k)}$ treating the complex variables $\boldsymbol{\lambda}^{(k)}$ and $\bar{\boldsymbol{\lambda}}^{(k)}$ as independent. The first steps are standard, see e.g. [27]. Using elementary linear algebra,

$$\begin{aligned} &|\Delta^{(N-1)}(\lambda_2,\dots,\lambda_N)|^2 \prod_{m=2}^{N} \omega(\lambda_m,\bar{\lambda}_m \mid \lambda_1,\bar{\lambda}_1) \\ = &\prod_{q=0}^{N-2} \langle P_q, Q_q\rangle \det_{2\le i,j\le N} \left(K_{11}^{(N-1)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j \mid \lambda_1,\bar{\lambda}_1)\right), \end{aligned} \tag{3.6}$$

where $K_{11}^{(N)}$ is the following kernel (of an integral operator):

$$K_{11}^{(N)}(x,\bar{x},y,\bar{y} \mid \lambda_1,\bar{\lambda}_1) = \sum_{k=0}^{N-1} \frac{\overline{P_k(x)}Q_k(y)}{\langle P_k, Q_k\rangle}\omega(x,\bar{x} \mid \lambda_1,\bar{\lambda}_1), \tag{3.7}$$

and $\{P_i,Q_i\}_{i=0}^{\infty}$ are holomorphic monic polynomials on $\mathbb{C}$, bi-orthogonal with respect to the weight $\omega(\cdot,\cdot \mid \lambda_1,\bar{\lambda}_1)$:

$$\langle P_i, Q_j\rangle := \int_{\mathbb{C}} dz d\bar{z} \omega(z,\bar{z} \mid \lambda_1,\bar{\lambda}_1)\overline{P_i(z)}Q_j(z) = \langle P_i, Q_i\rangle \delta_{i,j},\ 0\le i,j<\infty. \tag{3.8}$$

Notice that the bi-orthogonal polynomials depend on λ_1 and $\bar{\lambda}_1$ as parameters, but we will suppress this dependence in order to simplify the notation. We will establish the existence of the bi-orthogonal polynomials and the associated kernel (3.7) for the concrete weight ω by constructing them explicitly, for a general discussion see [1].

In what follows it will be convenient to define the reduced kernel $\kappa^{(N)}$ via

$$\begin{aligned} K_{11}^{(N)}(x,\bar{x},y,\bar{y} \mid \lambda_1,\bar{\lambda}_1) &= \kappa^{(N)}(\bar{x},y \mid \lambda_1,\bar{\lambda}_1)\omega(x,\bar{x} \mid \lambda_1,\bar{\lambda}_1), \\ \kappa^{(N)}(\bar{x},y \mid \lambda_1,\bar{\lambda}_1) &= \sum_{k=0}^{N-1} \frac{\overline{P_k(x)}Q_k(y)}{\langle P_k, Q_k\rangle}. \end{aligned} \tag{3.9}$$

Notice that the kernel $K_{11}^{(N)}$ is self-reproducing,

$$K_{11}^{(N)} * K_{11}^{(N)} = K_{11}^{(N)}.$$

(Equivalently, the corresponding integral operator acting on polynomials of degree N is a projection.) Therefore, Dyson's theorem is applicable to the calculation of the integral in (3.3). [3] Substituting (3.6) into (3.3) and applying the theorem, we find that

$$\begin{aligned} D_{11}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) &= \frac{\pi^{N-1}N!}{Z_N} \prod_{q=0}^{N-2} \langle P_q, Q_q\rangle \cdot e^{-|\lambda_1|^2} \\ &\quad \times \det_{2\le i,j\le k} \left(K_{11}^{(N-1)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j \mid \lambda_1,\bar{\lambda}_1)\right). \end{aligned} \tag{3.10}$$

Observe the emergence of the determinantal structure for the diagonal conditional overlaps.

[3] The self-adjointness of a kernel is not necessary for the applicability of Dyson's theorem.

The off-diagonal overlap $D_{12}^{(N,k)}$ as a function on $\mathbb{C}^{2k}$ can now be computed using Lemma 1:

$$\begin{aligned}
D_{12}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) &= -\frac{\pi^{N-1}N!}{Z_N}\frac{e^{-\lambda_{12}\bar\lambda_{12}-\lambda_1\bar\lambda_2}}{1-\lambda_{12}\bar\lambda_{12}}\hat{T}\left(\prod_{q=0}^{N-2}\langle P_q,Q_q\rangle\right. \\
&\times \det\left(\begin{array}{c|c} K_{11}^{(N-1)}(\lambda_2,\bar\lambda_1,\lambda_2,\bar\lambda_1\mid\lambda_1,\bar\lambda_2) & \begin{array}{c}K_{11}^{(N-1)}(\lambda_2,\bar\lambda_1,\lambda_j,\bar\lambda_j\mid\lambda_1,\bar\lambda_2),\\ 3\le j\le k\end{array} \\ \hline \begin{array}{c}K_{11}^{(N-1)}(\lambda_i,\bar\lambda_i,\lambda_2,\bar\lambda_1\mid\lambda_1,\bar\lambda_2),\\ 3\le i\le k\end{array} & \begin{array}{c}K_{11}^{(N-1)}(\lambda_i,\bar\lambda_i,\lambda_j,\bar\lambda_j\mid\lambda_1,\bar\lambda_2),\\ 3\le i,j\le k\end{array}\end{array}\right)\left.\vphantom{\prod_{q=0}^{N-2}}\right).
\end{aligned} \tag{3.11}$$

It is worth stressing that $\prod_{q=0}^{N-2}\langle P_q,Q_q\rangle$ is a function of $\lambda_1,\bar\lambda_1$, therefore the action of $\hat{T}$ on this product is non-trivial. Recall also that $\lambda_{ij}:=\lambda_i-\lambda_j$, $\bar\lambda_{ij}:=\bar\lambda_i-\bar\lambda_j$. The determinant in the above formula can be re-written using the following determinantal identity

$$\det_{1\le i,j\le n}(a_{ij}) = a_{11}\det_{2\le i,j\le n}\left(a_{11}^{-1}\det\begin{pmatrix} a_{11} & a_{1j}\\ a_{i1} & a_{ij}\end{pmatrix}\right),\ a_{11}\neq 0. \tag{3.12}$$

It follows from a well known identity for block determinants, see e.g. [31]:

$$\det\begin{pmatrix} A & B\\ C & D\end{pmatrix} = \det(A)\det\left(D-CA^{-1}B\right), \tag{3.13}$$

valid for invertible matrices A. Namely, choosing for $A=a_{11}\neq 0$ in (3.12) we have

$$\det_{2\le i,j\le N}\begin{pmatrix} a_{11} & a_{1j}\\ a_{i1} & a_{ij}\end{pmatrix} = \det(a_{11})\det_{2\le i,j\le N}\left(a_{ij}-a_{i1}a_{11}^{-1}a_{1j}\right) = a_{11}\det_{2\le i,j\le N}\left(a_{11}^{-1}(a_{11}a_{ij}-a_{i1}a_{1j})\right). \tag{3.14}$$

Eq. (3.12) can be seen as the simplest of Tanner's identities for determinants and Pfaffians, see e.g. [23] for a review. Applying (3.12) to (3.11) results into

$$\begin{aligned}
D_{12}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) &= -\frac{\pi^{N-2}N!}{Z_N}\hat{T}\left(\prod_{q=0}^{N-2}\langle P_q,Q_q\rangle\right)e^{-\lambda_1\bar\lambda_1-\lambda_2\bar\lambda_2}\kappa^{(N-1)}\left(\bar\lambda_1,\lambda_2|\lambda_1,\bar\lambda_2\right) \\
\times \det_{3\le i,j\le k}&\left(\frac{\omega(\lambda_i,\bar\lambda_i\mid\lambda_1,\bar\lambda_2)}{\kappa^{(N-1)}\left(\bar\lambda_1,\lambda_2|\lambda_1,\bar\lambda_2\right)}\det\begin{pmatrix}\kappa^{(N-1)}(\bar\lambda_1,\lambda_2\mid\lambda_1,\bar\lambda_2) & \kappa^{(N-1)}(\bar\lambda_1,\lambda_j\mid\lambda_1,\bar\lambda_2)\\ \kappa^{(N-1)}(\bar\lambda_i,\lambda_2\mid\lambda_1,\bar\lambda_2) & \kappa^{(N-1)}(\bar\lambda_i,\lambda_j\mid\lambda_1,\bar\lambda_2)\end{pmatrix}\right),
\end{aligned} \tag{3.15}$$

which explains the structure of the claim (2.22), (2.23) of Theorem 1. The final answers for the conditional overlaps are obtained by evaluating (3.10) and (3.15) on the real surface $\mathbb{C}^k\subset\mathbb{C}^{2k}$, specified by the equations $\boldsymbol{\lambda}^{(k)}=\overline{\bar{\boldsymbol{\lambda}}^{(k)}}$.

The proof of Theorem 1 is therefore reduced to the calculation of the reduced kernel $\kappa^{(N)}$ and the inner products of the bi-orthogonal polynomials $\langle P_q,Q_q\rangle$ for $q=0,1,2,\ldots$ The bi-orthogonal polynomials themselves are not the subject of our current investigation, therefore it is reasonable to follow the approach of [6] and derive expressions for $\kappa^{(N)}$ and $\langle P_q,Q_q\rangle$ directly in terms of the moment matrix M defined as

$$M_{ij}=\langle z^i,z^j\rangle,\ i,j\ge 0. \tag{3.16}$$

Let (L,D,U) be the LDU-decomposition of M. That is D is the diagonal matrix, L and U^T are the lower triangular matrices with the diagonal entries equal to 1 such that

$$M=LDU. \tag{3.17}$$

Therefore, $L^{-1}MU^{-1} = D$. Re-writing this identity in components we find that

$$\langle P_k, Q_l \rangle = D_{kk}\delta_{k,l}, \ \ k, l \geq 0, \tag{3.18}$$

where

$$\begin{aligned} P_k(z) &= \sum_{m=0}^{k} (\bar{L}^{-1})_{km} z^m, \\ Q_k(z) &= \sum_{m=0}^{k} z^m (U^{-1})_{mk}, \end{aligned} \tag{3.19}$$

for $k \geq 0$. We see that $\{P_q, Q_q\}_{k\geq 0}$ is the set of holomorphic monic polynomials bi-orthogonal with respect to the weight $\omega(\cdot, \cdot \mid \lambda_1, \bar{\lambda}_1)$. Comparing (3.18) with (3.8) we find that

$$\langle P_k, Q_k \rangle = D_{kk}, \ \ k \geq 0. \tag{3.20}$$

Substituting (3.20) and (3.19) into the expression (3.9) for the reduced kernel we also find that

$$\kappa^{(N)}(\bar{z}, z \mid \lambda_1, \bar{\lambda}_1) = \sum_{i,j=0}^{N-1} z^i C_{ij}^{(N-1)} \bar{z}^j, \tag{3.21}$$

where

$$C_{ij}^{(N)} = \sum_{k=0}^{N} (U^{-1})_{ik} \frac{1}{D_{kk}} (L^{-1})_{kj}, \ \ i, j \geq 0. \tag{3.22}$$

At least formally, the semi-infinite matrix $C^{(N)}$ converges to M^{-1} as $N \to \infty$. Perhaps less trivially, as a consequence of the Graham-Schmidt orthogonalisation procedure, it can be also characterised as the inverse of the $(N+1) \times (N+1)$ moment matrix $(\langle z^i, z^j \rangle)_{0 \leq i,j} \leq N$, see [6].

We are not attempting to justify the above formal operations with semi-infinite matrices in general, but in Section 3.4 a justification will be given for the integration weight at hand.

Now the proof of Theorem 1 has been reduced to the calculation of the LDU decomposition of of the moment matrix M.

Remark. We see that the expression for the off-diagonal overlap $D_{12}^{(N,k)}$ is determinantal with the kernel expressed as the 2×2 determinant of a matrix built out of the kernel corresponding to the weight $\omega(x, \bar{x} \mid \lambda, \bar{\lambda})$. Such a structure is to be expected from the general theory of orthogonal polynomials in the complex plane developed in [1]. Really, relation (3.4) can be re-written as

$$\begin{aligned} D_{12}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) &= -\frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{Z_N} \frac{N!}{(N-k)!} \int_{\mathbb{C}^{N-k}} \prod_{i=k+1}^{N} d\lambda_i d\bar{\lambda}_i |\Delta^{(N-2)}(\lambda_3, \lambda_4, \ldots, \lambda_N)|^2 \\ &\times \prod_{m=3}^{N} (\lambda_2 - \lambda_m)(\bar{\lambda}_1 - \bar{\lambda}_m) \pi\omega(\lambda_m, \bar{\lambda}_m \mid \lambda_1, \bar{\lambda}_2) \end{aligned}$$

By Dyson's theorem, the right hand side of this expression is proportional to the $(k-2) \times (k-2)$ determinant of the kernel associated with holomorphic polynomials, which are bi-orthogonal with respect to the weight

$$(u - z)(\bar{v} - \bar{z})\omega(z, \bar{z} \mid v, \bar{u}).$$

Such a kernel can be expressed in terms of a 2×2 determinant of the kernel associated with the weight $\omega(\cdot, \cdot \mid \lambda, \bar{\lambda})$, see formula (3.10) of [1], which can be considered as a generalisation of Christoffel's theorem for orthogonal polynomials in the complex plane. Our present calculation can be therefore regarded as a short re-derivation of the general expression of [1] in the particular context of integration weights associated with the overlaps. The main tools used in our calculation are the analyticity and determinant identities.

3.2 Lemma 1

It follows from (3.3) and (3.4) that both $D_{11}^{(N,k)}(\boldsymbol{\lambda}^{(k)})e^{\sum_{m=1}^{k}\lambda_k\bar{\lambda}_k}$ and $D_{12}^{(N,k)}(\boldsymbol{\lambda}^{(k)})e^{\sum_{m=1}^{k}\lambda_k\bar{\lambda}_k}$ are polynomials in $\boldsymbol{\lambda}^{(k)}, \bar{\boldsymbol{\lambda}}^{(k)}$. Therefore $D_{11}^{(N,k)}$ and $D_{12}^{(N,k)}$ are entire functions on $\mathbb{C}^{2k}$.

Recall the definition of the transposition $\hat{T}$ acting on functions on $\mathbb{C}^{2k}$:

$$\hat{T}f(\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \ldots) = f(\lambda_1, \bar{\lambda}_2, \lambda_2, \bar{\lambda}_1, \ldots) \tag{3.23}$$

Comparing (3.3) and (3.4), we see that for $k \geq 2$,

$$D_{12}^{(N,k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{e^{-|\lambda_1-\lambda_2|^2}}{1-|\lambda_1-\lambda_2|^2}\hat{T}D_{11}^{(N,k)}(\boldsymbol{\lambda}^{(k)}), \tag{3.24}$$

for any $(\boldsymbol{\lambda}^{(k)}, \bar{\boldsymbol{\lambda}}^{(k)}) \in \mathbb{C}^{2k}$. Lemma 1 is proved.

3.3 Heuristic derivation of $N = \infty$ results in the bulk assuming T-invariance

The task of calculating bi-orthogonal polynomials (3.8) is considerably simpler at the special point $\lambda_1 = \bar{\lambda}_1 = 0$. In this case the weight function reduces to

$$\omega(z, \bar{z} \mid 0, 0) = \frac{1}{\pi}(1+|z|^2)e^{-|z|^2}, \tag{3.25}$$

which is an $U(1)$-invariant function. The bi-orthogonal polynomials associated with $U(1)$-invariant weights are just the monomials,

$$P_k(x) = Q_k(x) = x^k, \;\; k \geq 0. \tag{3.26}$$

Their inner products can also be computed explicitly,

$$\langle P_k, Q_k \rangle = k!(k+2), \;\; k \geq 0, \tag{3.27}$$

leading to the following kernel:

$$K_{11}^{(N)}(x, \bar{x}, y, \bar{y} \mid 0, 0) = \frac{1}{\pi}(1+|x|^2)e^{-|x|^2}\sum_{k=0}^{N-1}\frac{(\bar{x}y)^k}{(k+2)k!}. \tag{3.28}$$

As $N \to \infty$, the limiting kernel in the bulk is

$$K_{11}^{(bulk)}(x, \bar{x}, y, \bar{y} \mid 0, 0) \;\; = \;\; \frac{1}{\pi}(1+|x|^2)e^{-|x|^2}\kappa^{(bulk)}(\bar{x}, y \mid 0, 0), \tag{3.29}$$

where

$$\kappa^{(bulk)}(\bar{x}, y \mid 0, 0) = \frac{1}{(\bar{x}y)^2} + \left(\frac{1}{(\bar{x}y)} - \frac{1}{(\bar{x}y)^2}\right)e^{\bar{x}y}. \tag{3.30}$$

Alternatively, we can write

$$\kappa^{(bulk)}(\bar{x}, y \mid 0, 0) = \frac{d}{dz}\left(\frac{e^z - 1}{z}\right)\Big|_{z=\bar{x}y}. \tag{3.31}$$

The N-dependent pre-factor in the right hand side of (3.10) is N/π, which leads to the following answer for the conditional overlap in the bulk:

$$D_{11}^{(bulk,\ k)}(0, \lambda_2, \ldots, \lambda_k) = \frac{1}{\pi} \det_{2 \le i,j \le k}\left(K_{11}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid 0, 0)\right) \tag{3.32}$$

Let us *assume* the extended translational invariance for the diagonal overlaps regarded as functions on C^{2k}, which means that $D_{11}^{(bulk,\ k)}$ is invariant under the shift $\lambda_m \to \lambda_m + \epsilon$, $\bar{\lambda}_m \to \bar{\lambda}_m + \bar{\epsilon}$, $m = 1, 2, \ldots, k$, where $\epsilon, \bar{\epsilon}$ are independent complex variables. Then

$$\begin{aligned} D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) &= D_{11}^{(bulk,\ k)}(0, \lambda_2 - \lambda_1, \ldots, \lambda_k - \lambda_1) \\ &= \frac{1}{\pi} \det_{2 \le i,j \le k}\left(K_{11}^{(bulk)}(\lambda_i - \lambda_1, \bar{\lambda}_i - \bar{\lambda}_1, \lambda_j - \lambda_1, \bar{\lambda}_j - \bar{\lambda}_1 \mid 0, 0)\right). \end{aligned} \tag{3.33}$$

We conclude that

$$D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = \frac{1}{\pi} \det_{2 \le i,j \le k}\left(K_{11}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1)\right), \tag{3.34}$$

where

$$K_{11}^{(bulk)}(x, \bar{x}, y, \bar{y} \mid \lambda, \bar{\lambda}) = \frac{1}{\pi}(1 + |x - \lambda|^2) e^{-|x-\lambda|^2} \kappa^{(bulk)}(\bar{x}, y \mid \lambda, \bar{\lambda}), \tag{3.35}$$

and the reduced kernel is

$$\kappa^{(bulk)}(\bar{x}, y \mid \lambda, \bar{\lambda}) = \frac{d}{dz}\left(\frac{e^z - 1}{z}\right)\Big|_{z=(\bar{x}-\bar{\lambda})(y-\lambda)}, \tag{3.36}$$

which agrees with the statement (2.28) of Corollary 1.

To calculate the off-diagonal conditional overlaps, let us *assume* that the relation (2.14) remains valid at $N = \infty$ as well. Then

$$D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{1}{\pi} \frac{e^{-|\lambda_1 - \lambda_2|^2}}{1 - |\lambda_1 - \lambda_2|^2} \hat{T} \det_{2 \le i,j \le k}\left(K_{11}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1)\right). \tag{3.37}$$

Applying the determinant identity (3.12), we find

$$D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{1}{\pi^2} \kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) \det_{3 \le i,j \le k}\left(K_{12}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_2)\right), \tag{3.38}$$

where

$$\begin{aligned} K_{12}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_2) &= \frac{\omega^{(bulk)}(\lambda_i, \bar{\lambda}_i \mid \lambda_1, \bar{\lambda}_2)}{\kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2)} \\ &\times \det\begin{pmatrix} \kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) & \kappa^{(bulk)}(\bar{\lambda}_1, \lambda_j \mid \lambda_1, \bar{\lambda}_2) \\ \kappa^{(bulk)}(\bar{\lambda}_i, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) & \kappa^{(bulk)}(\bar{\lambda}_i, \lambda_j \mid \lambda_1, \bar{\lambda}_2) \end{pmatrix}, \end{aligned} \tag{3.39}$$

which agrees with the statement (2.32) of Corollary 1.

The above calculation is rather simple, but non-rigorous - it rests on the assumptions of the extended translational invariance of conditional overlaps in the bulk and the validity of Lemma 1 at $N = \infty$. We could try justifying these assumptions using analysis, but as it turns out, it is possible to obtain a fairly simple explicit expression for the kernel at $N < \infty$, thus enabling the study of conditional overlaps not only in the bulk of the spectrum, but also near the spectral edge. Notice that in the latter case the translational invariance is absent in principle.

3.4 The kernel for $N < \infty$

3.4.1 The LDU decomposition of the moment matrix.

We will use the relation between the kernel and the moment matrix established in Section 3.1. An explicit computation of $\langle z^i, z^j \rangle$ with the weight $\omega(\cdot,\cdot\,|,\lambda,\bar{\lambda})$ defined (3.5) gives

$$M_{ij} = i!\left[\delta_{ij}\left((1+\lambda\bar{\lambda})+(i+1)\right) - \delta_{i+1,j}\lambda(i+1) - \delta_{i,j+1}\bar{\lambda}\right],\ i,j \geq 0. \tag{3.40}$$

Crucially, the moment matrix is tri-diagonal, which makes explicit calculations leading to the kernel possible. The recursive formulae for computing the LDU decomposition and the inverse of a tri-diagonal matrix are well-known. What makes our case special however, is that the recursions we get can be solved exactly in terms of the exponential polynomials. At some point it would be interesting to understand the algebraic reasons for the exact solvability of our problem, but in the mean time we adopt a tour de force approach.

Let μ be the following tri-dagonal matrix:

$$\mu_{ij} = \delta_{ij}\left((1+\lambda\bar{\lambda})+(i+1)\right) - \delta_{i+1,j}\lambda(i+1) - \delta_{i,j+1}\bar{\lambda},\ i,j \geq 0. \tag{3.41}$$

As M is the product of of μ and the diagonal matrix with entries $i!$, the LDU decomposition of M is easy to construct from the LDU decomposition of μ. If

$$\mu = LDU, \tag{3.42}$$

where $D_{pq} = d_p\delta_{pq}, L_{pq} = \delta_{pq} + l_p\delta_{p,q+1}, U_{pq} = \delta_{pq} + u_q\delta_{q,p+1},\ p,q \geq 0$, then

$$u_{p+1} = -\frac{(1+p)\lambda}{d_p},\ p \geq 0, \tag{3.43}$$

$$l_{p+1} = -\frac{\bar{\lambda}}{d_p},\ p \geq 0, \tag{3.44}$$

$$d_p = -d_{p-1}l_pu_p\mathbb{1}_{p\geq 1} + 2 + \lambda\bar{\lambda} + p,\ p \geq 0, \tag{3.45}$$

defining $d_{-1} \equiv 0$. Let $x = \lambda\bar{\lambda}$. To determine the LDU decomposition of μ we have to solve the first order non-linear recursion for d_p's:

$$\begin{aligned} d_p &= 2 + x + p - \frac{px}{d_{p-1}},\ p \geq 1, \\ d_0 &= 2 + x. \end{aligned} \tag{3.46}$$

This recursion can be linearised via the substitution $d_p = \frac{r_{p+1}}{r_p}$, which, upon choosing $r_0 = 1$, gives

$$\begin{aligned} r_{p+1} + pxr_{p-1} &= (2+x+p)r_p,\ p \geq 1, \\ r_1 &= 2 + x. \end{aligned} \tag{3.47}$$

The unique solution of (3.47) is

$$r_p(x) = p!\sum_{m=0}^{p}\frac{(p+1-m)}{m!}x^m = (p+1)!e_p(x) - p!xe_{p-1}(x), \tag{3.48}$$

where $e_p(x) = \sum_{k=0}^{p}\frac{x^p}{p!}$ is the exponential polynomial of degree p.

Therefore,

$$r_p(x) = p! f_p(x), \;\; p = 0, 1, \ldots, \tag{3.49}$$

where f_p's are the polynomials defined in (2.16). Converting the LDU decomposition of μ to the LDU decomposition of M and updating notations, we find that $M = LDU$, where

$$L_{pm} = \delta_{pm} - \bar{\lambda}\frac{f_{p-1}(\lambda\bar{\lambda})}{f_p(\lambda\bar{\lambda})}\delta_{p,m+1}, \;\; p, m \geq 0, \tag{3.50}$$

$$D_{mm} = (m+1)!\frac{f_{m+1}(\lambda\bar{\lambda})}{f_m(\lambda\bar{\lambda})}, \;\; m \geq 0, \tag{3.51}$$

$$U_{mq} = \delta_{mq} - \lambda\frac{f_{m-1}(\lambda\bar{\lambda})}{f_m(\lambda\bar{\lambda})}\delta_{q,m+1}, \;\; m, q \geq 0. \tag{3.52}$$

Using the relation (3.18) between the LDU decomposition and the inner products of the bi-orthogonal polynomials, we conclude that

$$\langle P_p, Q_p \rangle = (p+1)!\frac{(p+2)e_{p+1}(x) - xe_p(x)}{(p+1)e_p(x) - xe_{p-1}(x)}, \tag{3.53}$$

which coincides with (3.27) at the point $x = 0$, as it should.

3.4.2 The inner products of the bi-orthogonal polynomials and the pre-factor in (3.10)

Now we can calculate the factor in front of the determinant in the r. h. s. of (3.10). Using the relation (3.53) we find

$$\prod_{q=0}^{N-2}\langle P_q, Q_q \rangle = \prod_{q=0}^{N-2} D_{qq} = \prod_{q=0}^{N-2} q! \cdot r_{N-1}(\lambda\bar{\lambda}) \tag{3.54}$$

Therefore,

$$\pi^{N-1}\frac{N!}{Z_N}\prod_{q=0}^{N-2}\langle P_q, Q_q \rangle \cdot e^{-\lambda\bar{\lambda}} = \frac{f_{N-1}(\lambda\bar{\lambda})}{\pi}e^{-\lambda\bar{\lambda}}, \tag{3.55}$$

which allows us to make the operation of $\hat{T}$ on the inner product explicit.

3.4.3 Inversion of the L and U factors and the kernel

The inverse of the lower-triangular matrix L (resp. upper triangular matrix U) is a lower (resp. upper) triangular matrix. The corresponding matrix elements can be computed directly from the relations $LL^{-1} = I$, $UU^{-1} = I$ using the explicit expressions (3.50) for the decomposition factors. The answer is

$$(L^{-1})_{pq} = \begin{cases} 0 & q > p, \\ 1 & q = p, \\ \bar{\lambda}^{p-q}\frac{f_q(x)}{f_p(x)} & q < p, \end{cases} \qquad (U^{-1})_{pq} = \begin{cases} \lambda^{q-p}\frac{f_p(x)}{f_q(x)} & q > p, \\ 1 & q = p, \\ 0 & q < p. \end{cases} \tag{3.56}$$

Substituting (3.56) and (3.51) into the formula (3.21) we find that

$$\kappa^{(N+1)}(\bar{\mu}, \nu \mid \lambda, \bar{\lambda}) = G^{(N)}\left(\lambda\bar{\lambda}, \frac{\bar{\mu}}{\bar{\lambda}}, \frac{\nu}{\lambda}\right), \tag{3.57}$$

where

$$G^{(N)}(x, y, z) = \sum_{m,n=0}^{N} f_m(x) f_n(x) y^m z^n \sum_{k=m\vee n}^{N}\frac{x^k}{(k+1)! f_k(x) f_{k+1}(x)} \tag{3.58}$$

is a function on $\mathbb{C}^3$ and $a \vee b := max(a, b)$.

3.4.4 Simplification of the reduced kernel for $N < \infty$.

The above form of the reduced kernel is not well suited for studying the large-N asymptotic of the overlaps. In particular, we do not see how to calculate the large-N limit of the kernel directly from (3.57). Fortunately, it can be considerably simplified via a sequence of lucky cancellations yielding formula (2.21).

The inner sum in (3.57) can be simplified as follows: Let $\Phi_n : \mathbb{C} \to \mathbb{C}$ be such that

$$\Phi_n(x) := \sum_{k=0}^{n} \frac{x^k}{(k+1)! f_k(x) f_{k+1}(x)}, \tag{3.59}$$

where we define $\Phi_{-1} \equiv 0$. Then

$$G^{(N)}(x, y, z) = \sum_{m,n=0}^{N} f_m(x) f_n(x) y^m z^n \left(\Phi_N(x) - \Phi_{m \vee n-1}(x)\right). \tag{3.60}$$

We have the following key technical result:

Lemma 2.

$$\Phi_n(x) = \frac{(n+2-x)}{x^2 f_{n+1}(x)} + \frac{x-1}{x^2}, \;\; n = 0, 1, \ldots \tag{3.61}$$

Proof. For a fixed value of x, the sequence

$$\Phi_n(x) = \sum_{k=0}^{n} \frac{x^k}{(k+1)! f_k(x) f_{k+1}(x)} \tag{3.62}$$

satisfies the following difference equation:

$$\Phi_{n+1}(x) = \Phi_n(x) + \frac{x^{n+1}}{(n+2)! f_{n+1}(x) f_{n+2}(x)}, \tag{3.63}$$

$$\Phi_0(x) = \frac{1}{2+x}. \tag{3.64}$$

Using $f_1(x) = 2 + x$, it is easy to check that the expression (3.61) satisfies the initial condition (3.64). Assuming that Φ_n is given by (3.61), we find from the equation (3.63) that

$$\Phi_{n+1} = \frac{x-1}{x^2} + \frac{(n+2-x) f_{n+2}(x) + \frac{x^{n+3}}{(n+2)!}}{x^2 f_{n+1}(x) f_{n+2}(x)}. \tag{3.65}$$

A direct calculation based on the definitions (2.15) for the exponential polynomials e_n and (2.16) for the polynomials f_n confirms that

$$(n+2-x) f_{n+2}(x) + \frac{x^{n+3}}{(n+2)!} = (n+3-x) f_{n+1}(x). \tag{3.66}$$

Therefore,

$$\Phi_{n+1} = \frac{x-1}{x^2} + \frac{(n+3-x)}{x^2 f_{n+2}(x)}, \tag{3.67}$$

and Lemma 2 is proved by induction. □

Substituting (3.61) into (3.60), we find

$$G^{(N)}(x,y,z) = \frac{(N+2-x)}{x^2 f_{N+1}(x)} \left(\sum_{m=0}^{N} f_m(x) y^m\right)\left(\sum_{n=0}^{N} f_n(x) z^n\right) + \frac{1}{x^2}\sum_{m,n=0}^{N}(x - m\vee n - 1)\frac{f_m(x)f_n(x)}{f_{m\vee n}(x)} y^m z^n. \quad (3.68)$$

Let

$$\alpha_n(x,y) := \sum_{m=0}^{n} f_m(x) y^m, m = 0, 1, \ldots. \quad (3.69)$$

Then, the first term in the r.h.s. of (3.68) is equal to

$$\frac{(N+2-x)}{x^2 f_{N+1}(x)}\alpha_N(x,y)\alpha_N(x,z). \quad (3.70)$$

The second term can be also be expressed in terms of α_n's:

$$\frac{1}{x^2}\sum_{m,n=0}^{N}(x - m\vee n - 1)\frac{f_m(x)f_n(x)}{f_{m\vee n}(x)} y^m z^n = \frac{1}{x^2}\sum_{n=0}^{N}(x-n-1)f_n(x)(yz)^n$$
$$+\frac{1}{x^2}\sum_{m>n\geq 0}^{N}(x-m-1)f_n(x)y^m z^n + \frac{1}{x^2}\sum_{0\leq m<n}^{N}(x-n-1)f_m(x)y^m z^n$$

$$= \frac{1}{x^2}\left(\left(x - \omega\frac{\partial}{\partial\omega} - 1\right)\alpha_N(x,\omega)\mid_{\omega=yz} + \psi_x(y,z) + \psi_x(z,y)\right), \quad (3.71)$$

where

$$\psi_x(y,z) = \sum_{n>m\geq 0}^{N}(x-n-1)f_m(x)y^m z^n. \quad (3.72)$$

Next,

$$\begin{aligned}
\psi_x(y,z) &= \left(x - z\frac{\partial}{\partial z} - 1\right)\sum_{n>m\geq 0}^{N} f_m(x)y^m z^n \\
&= \left(x - z\frac{\partial}{\partial z} - 1\right)\sum_{m=0}^{N} f_m(x)y^m \sum_{n=m+1}^{N} z^n \\
&= \left(x - z\frac{\partial}{\partial z} - 1\right)\sum_{m=0}^{N} f_m(x)y^m \left(\frac{z^{N+1} - z^{m+1}}{z-1}\right) \\
&= \left(x - z\frac{\partial}{\partial z} - 1\right)\frac{1}{(z-1)}\left(z^{N+1}\alpha_N(x,y) - z\alpha_N(x,yz)\right). \quad (3.73)
\end{aligned}$$

Substituting (3.73) into (3.71) and then substituting the result and (3.70) into (3.68), we find that

$$\begin{aligned}
G^{(N)}(x,y,z) &= \frac{(N+2-x)}{x^2 f_{N+1}(x)}\alpha_N(x,y)\alpha_N(x,z) + \frac{1}{x^2}\left(x - \omega\frac{\partial}{\partial\omega} - 1\right)\alpha_N(x,\omega)\mid_{\omega=yz} \\
&+\frac{1}{x^2}\left(x - z\frac{\partial}{\partial z} - 1\right)\frac{z}{z-1}\left(z^N\alpha_N(x,y) - \alpha_N(x,yz)\right) \\
&+\frac{1}{x^2}\left(x - y\frac{\partial}{\partial y} - 1\right)\frac{y}{y-1}\left(y^N\alpha_N(x,z) - \alpha_N(x,yz)\right). \quad (3.74)
\end{aligned}$$

To simplify the expression for $G^{(N)}$ further, we need an expression for α_N in terms of the exponential polynomials:

$$\begin{aligned}
\alpha_N(x,y) &= \sum_{n=0}^{N} f_n(x)y^n = \sum_{n=0}^{N} \left((n+1)e_n(x) - xe_{n-1}(x)\right) y^n \\
&= \left(\frac{\partial}{\partial y}y - xy\right)\sum_{n=0}^{N} e_n(x)y^n + xy^{N+1}e_N(x) \\
&= \left(\frac{\partial}{\partial y}y - xy\right)\frac{e_N(yx) - y^{N+1}e_N(x)}{1-y} + xy^{N+1}e_N(x). \qquad (3.75)
\end{aligned}$$

Explicitly,

$$\alpha_N(x,y) = \frac{e_N(yx)}{(1-y)^2} - \frac{yx}{(1-y)}\frac{(yx)^N}{N!} - ((N+2-x)-(N+1-x)y)\frac{y^{N+1}e_N(x)}{(1-y)^2}. \qquad (3.76)$$

Substituting (3.76) into (3.74), computing the derivatives and grouping the terms according to the denominators we arrive at

$$x^2G^{(N)}(x,y,z) = \frac{T_A^{(N)}(x,y,z)}{(1-y)^2(1-z)^2} + \frac{T_B^{(N)}(x,y,z)}{(1-y)(1-z)} + \frac{T_C^{(N)}(x,y,z)}{(1-y)^2(1-z)} + \frac{T_C^{(N)}(x,z,y)}{(1-z)^2(1-y)}, \quad (3.77)$$

where

$$\begin{aligned}
T_A^{(N)}(x,y,z) &= \frac{(N+2-x)}{f_{N+1}(x)}e_N(yx)e_N(zx) - e_N(xyz) \qquad (3.78)\\
&\quad + \frac{(N+2)}{f_{N+1}(x)(N+1)!}\left((zx)^{N+1}e_N(yx) + (yx)^{N+1}e_N(zx) - (xyz)^{N+1}e_N(x)\right),
\end{aligned}$$

$$\begin{aligned}
T_B^{(N)}(x,y,z) &= \frac{(N+2-x)}{f_{N+1}(x)}\frac{(yx)^{N+1}}{N!}\frac{(zx)^{N+1}}{N!} + xe_N(xyz) + (N+1-x)\frac{(xyz)^{N+1}}{N!} \qquad (3.79)\\
&\quad - \frac{(N+2)(N+1-x)}{f_{N+1}(x)(N+1)!} \\
&\quad \times \left(\frac{(zx)^{N+1}(yx)^{N+1}}{N!} + \frac{(yx)^{N+1}(zx)^{N+1}}{N!} + (N+1-x)(xyz)^{N+1}e_N(x)\right),
\end{aligned}$$

$$\begin{aligned}
T_C^{(N)}(x,y,z) &= -\frac{(N+2-x)}{f_{N+1}(x)}e_N(yx)\frac{(zx)^{N+1}}{N!} + \frac{(xyz)^{N+1}}{N!} + \frac{(N+2)}{f_{N+1}(x)(N+1)!} \qquad (3.80)\\
&\quad \times \left((N+1-x)(zx)^{N+1}e_N(yx) - \frac{(yx)^{N+1}(zx)^{N+1}}{N!} - (N+1-x)e_N(x)(xyz)^{N+1}\right).
\end{aligned}$$

A straightforward simplification of each of the T-terms gives:

$$\begin{aligned}
f_{N+1}(x)T_A^{(N)}(x,y,z) &= (N+2)(e_{N+1}(yx)e_{N+1}(zx) - e_{N+1}(xyz)e_{N+1}(x)) \qquad (3.81)\\
&\quad -x(e_N(yx)e_N(zx) - e_N(xyz)e_N(x)),
\end{aligned}$$

$$\begin{aligned}
f_{N+1}(x)T_B^{(N)}(x,y,z) &= x\frac{(yx)^{N+1}(zx)^{N+1}}{N!(N+1)!} + x(N+1-x)e_N(x)\frac{(xyz)^{N+1}}{(N+1)!} \qquad (3.82)\\
&\quad +xe_N(xyz)\left((N+2-x)e_N(x) + (N+2)\frac{x^{N+1}}{(N+1)!}\right),
\end{aligned}$$

$$f_{N+1}(x)T_C^{(N)}(x,y,z) = x\frac{(xyz)^{N+1}}{(N+1)!}e_N(x) - xe_N(yx)\frac{(zx)^{N+1}}{(N+1)!}. \tag{3.83}$$

Substituting (3.81), (3.82) and (3.83) into (3.77) we arrive at

$$\begin{aligned} x^2 f_{N+1}(x)G^{(N)}(x,y,z) &= \frac{(N+2)W_{N+1}(x,y,z) - xW_N(x,y,z)}{(1-y)^2(1-z)^2} \\ &+x\frac{(xyz)^{N+1}e_N(x) - (xz)^{N+1}e_N(xy)}{(N+1)!(1-y)^2(1-z)} \\ &+x\frac{(xyz)^{N+1}e_N(x) - (xy)^{N+1}e_N(xz)}{(N+1)!(1-z)^2(1-y)} \\ &-x\frac{(xyz)^{N+1}}{(N+1)!}\frac{e_{N+1}(x)+xe_N(x)}{(1-y)(1-z)}, \end{aligned} \tag{3.84}$$

where

$$W_N(x,y,z) := e_N(xy)e_N(xz) - (1-x(1-y)(1-z))e_N(xyz)e_N(x), \;\; N \in \mathbb{N}. \tag{3.85}$$

This answer is already well-suited for the calculation of the kernel for the bulk and the edge scaling limits of the overlaps, but it still looks rather complicated. Fortunately, it can be re-written in a shorter form: Observing that $e_n(x) = e_{n+1}(x) - \frac{x^{n+1}}{(n+1)!}$, we find that

$$\begin{aligned} W_N(x,y,z) &= W_{N+1}(x,y,z) - e_{N+1}(xy)\frac{(xz)^{N+1}}{(N+1)!} - e_{N+1}(xz)\frac{(xy)^{N+1}}{(N+1)!} \\ &+ \left(e_{N+1}(xyz)\frac{(x)^{N+1}}{(N+1)!} + e_{N+1}(x)\frac{(xyz)^{N+1}}{(N+1)!}\right)(1-x(1-y)(1-z)) \\ &+x(1-y)(1-z)\frac{(yzx^2)^{N+1}}{((N+1)!)^2}. \end{aligned} \tag{3.86}$$

Expressing W_{N+1} and W_N in (3.84) in terms of W_{N+2} and W_{N+1}, using (3.86) one finds

$$\begin{aligned} x^2 f_{N+1}(x)G^{(N)}(x,y,z) &= \frac{(N+2)W_{N+2}(x,y,z) - xW_{N+1}(x,y,z)}{(1-y)^2(1-z)^2} \\ &-\frac{x}{(1-y)(1-z)}\frac{(xyz)^{N+2}}{(N+1)!}e_{N+2}(x). \end{aligned} \tag{3.87}$$

It is straightforward to check that

$$-x(1-y)(1-z)\frac{(xyz)^{N+2}}{(N+1)!}e_{N+2}(x) = (N+2)H_{N+2}(x,y,z) - xH_{N+1}(x,y,z), \tag{3.88}$$

where

$$H_N(x,y,z) := \frac{(1-y)(1-z)}{N!}\frac{x^{N+1}e_N(xyz) - (xyz)^{N+1}e_N(x)}{(yz-1)}. \tag{3.89}$$

It follows from (3.87, 3.88), that

$$x^2 f_{N+1}(x)G^{(N)}(x,y,z) = \frac{(N+2)\mathfrak{F}_{N+2}(x,y,z) - x\mathfrak{F}_{N+1}(x,y,z)}{(1-y)^2(1-z)^2}, \tag{3.90}$$

where

$$\mathfrak{F}_n(x,y,z) := W_n(x,y,z) + H_n(x,y,z), n \;\in \mathbb{N}, \tag{3.91}$$

is a function on $\mathbb{C}^3$ defined in (2.17). Finally, substituting (3.90) into (3.57) we arrive at the expression (2.21) for the reduced kernel. Theorem 1 is proved.

Remark. The final part of the proof following (3.84) is not very satisfying as it is both non-obvious and reliant on unexpected cancellations. A more direct route to (3.90) is to substitute the partition of unity $1 = \mathbb{1}_{m<n} + \mathbb{1}_{m\geq n}$ into the double sum in (3.60), represent the indicator functions as a contour integral,

$$\mathbb{1}_{m<n} = \oint \frac{dz}{2\pi i} \frac{z^{m-n}}{1-z},$$

and analyse the resulting expression for $G_N(x, y, z)$ as a sum of two contour integrals. The integrands of each of the integrals contain poles of order N as well as poles of order 1 and 2. It turns out that the contributions from the high order poles cancel, and the sum of contributions from the poles of low order gives (3.90), see [32] for details.

3.5 The proof of Corollary 1

The proof is based on the following elementary remark: for any fixed $x \in \mathbb{C}$

$$\lim_{N\to\infty} e_N(x) = e^x. \tag{3.92}$$

Consequently,

$$\frac{f_N(x)}{N} = (1+N^{-1})e_N(x) - N^{-1}xe_{N-1}(x) \stackrel{N\uparrow\infty}{\longrightarrow} e^x. \tag{3.93}$$

Therefore, the factor in front of the determinant in the r.h.s. of (2.18) divided by N converges for $|\lambda_1|^2 < N$ to

$$\lim_{N\to\infty} \frac{f_{N-1}(|\lambda_1|^2)}{N\pi} e^{-|\lambda_1|^2} = \frac{1}{\pi}, \tag{3.94}$$

as is well known from the Ginibre ensemble, cf. [22]. The large-N limit of the reduced kernel defined in (2.21) is most easily taken when fixing all arguments of the kernel, that is remaining in the vicinity of the origin, as the spectral edge is located at $\sqrt{N}$. The same bulk limit close to the origin was taken already in Ginibre's original paper for the complex eigenvalue correlations [19]. For our kernel we thus have

$$\begin{aligned}
\lim_{N\to\infty} \kappa^{(N)}(\bar{x}, y \mid \lambda, \bar{\lambda}) &= e^{-\lambda\bar{\lambda}} \frac{\lim_{N\to\infty} \mathfrak{F}_{N+1}\left(\lambda\bar{\lambda}, \frac{\bar{x}}{\bar{\lambda}}, \frac{y}{\lambda}\right)}{(\bar{x}-\bar{\lambda})^2(y-\lambda)^2} \\
&= \frac{e^{\bar{x}y}}{(\bar{x}-\bar{\lambda})^2(y-\lambda)^2}\left(e^{-(\bar{x}-\bar{\lambda})(y-\lambda)} - 1 + (\bar{x}-\bar{\lambda})(y-\lambda)\right) \\
&= \frac{e^{\bar{x}y}}{(\bar{x}-\bar{\lambda})^2(y-\lambda)^2} e^{(2)}(-(\bar{x}-\bar{\lambda})(y-\lambda)),
\end{aligned} \tag{3.95}$$

where $e^{(m)}(x) := \sum_{n=m}^{\infty} \frac{x^n}{n!}$, $m = 0, 1, \ldots$ In the second equality we used the definition (2.17) of the polynomials $\mathfrak{F}_N$. Therefore, the bulk scaling limit of the kernel $K_{11}^{(N)}$ defined in (2.19) is

$$\begin{aligned}
\lim_{N\to\infty} K_{11}^{(N)}(x, \bar{x}, y, \bar{y} \mid \lambda, \bar{\lambda}) &= \frac{e^{-x\bar{x}+y\bar{x}}}{\pi}\left(1 + (x-\lambda)(\bar{x}-\bar{\lambda})\right)\frac{e^{(2)}(-(\bar{x}-\bar{\lambda})(y-\lambda))}{(\bar{x}-\bar{\lambda})^2(y-\lambda)^2} \\
&= e^{(y-x)\bar{\lambda}}\frac{1}{\pi}(1+(x-\lambda)(\bar{x}-\bar{\lambda}))e^{-(x-\lambda)(\bar{x}-\bar{\lambda})} \\
&\quad \times \frac{d}{dz}\left(\frac{e^z-1}{z}\right)\bigg|_{z=(\bar{x}-\bar{\lambda})(y-\lambda)}.
\end{aligned} \tag{3.96}$$

Notice that the factor $e^{(y-x)\bar{\lambda}}$ in the r.h.s. of (3.96) corresponds to the conjugation of the kernel $K(\lambda_i, \lambda_j) \to \phi(\lambda_i) K(\lambda_i, \lambda_j) \phi^{-1}(\lambda_j)$, which does not change the value of the determinant in the expression (2.18) for the conditional overlap. This remark allows us to write the bulk scaling limit of the kernel as

$$K_{11}^{(bulk)}(x, \bar{x}, y, \bar{y} \mid \lambda, \bar{\lambda}) = \frac{1}{\pi} 1 + (x-\lambda)(\bar{x}-\bar{\lambda}) e^{-(x-\lambda)(\bar{x}-\bar{\lambda})} \frac{d}{dz}\left(\frac{e^z - 1}{z}\right)\Big|_{z=(\bar{x}-\bar{\lambda})(y-\lambda)}. \tag{3.97}$$

Expressions (3.94), (3.97) solve the problem of computing the large-N limit of (2.18), thus proving claims (2.28), (2.29), (2.30) and (2.31) of Corollary 1.

It remains to calculate the bulk scaling limit of $D_{12}^{(N,k)}$ starting with its determinantal representation (2.22). Using (3.95) and (3.96), one finds that the large-N limit of the pre-factor in (2.22) divided by N is

$$\lim_{N\to\infty} \frac{e^{-|\lambda_1|^2 - |\lambda_2|^2}}{N\pi^2} f_{N-1}(\lambda_1 \bar{\lambda}_2) \kappa^{(N-1)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) = \frac{\kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2)}{\pi^2}, \tag{3.98}$$

where $\kappa^{(bulk)}$ is defined in (2.31), and

$$\begin{aligned}\lim_{N\to\infty} K_{12}^{(N-1)}(x, \bar{x}, y, \bar{y} \mid u, \bar{u}, v, \bar{v}) &= e^{(y-x)\bar{v}} \frac{\omega^{(bulk)}(x, \bar{x} \mid u, \bar{v})}{\kappa^{(bulk)}(\bar{u}, v \mid u, \bar{v})} \\ &\times \det \begin{pmatrix} \kappa^{(bulk)}(\bar{u}, v \mid u, \bar{v}) & \kappa^{(bulk)}(\bar{u}, y \mid u, \bar{v}) \\ \kappa^{(bulk)}(\bar{x}, v \mid u, \bar{v}) & \kappa^{(bulk)}(\bar{x}, y \mid u, \bar{v}) \end{pmatrix}, \end{aligned} \tag{3.99}$$

where the weight $\omega^{(bulk)}$ is defined by (2.30). Calculating the large-N limit of (2.22) with the help of (3.98) and (3.99), and using the fact that conjugation of the kernel by $e^{\lambda_i \bar{\lambda}_2}$ does not change the determinant, we arrive at the characterisation (2.32), (2.33) for the bulk scaling limit of the off-diagonal conditional overlaps. Corollary 1 is proved.

3.6 Corollary 2

The calculation is based on the following two asymptotic formulae: Let us fix $a, b \in \mathbb{C}$. Then

$$\begin{aligned}\log \frac{(N + \sqrt{N}a + b)^{N+1}}{(N+1)!} &= (N + \sqrt{N}a + b) - \frac{1}{2}\log 2\pi N \\ &\quad -\frac{a^2}{2} + \frac{a - ab + \frac{1}{3}a^3}{\sqrt{N}} + O(N^{-1}),\end{aligned} \tag{3.100}$$

$$e_{N+k}(N + \sqrt{N}a + b) = e^{N+\sqrt{N}a+b}\left(F(a) + \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi N}}\left(\frac{a^2}{3} - b + k + \frac{2}{3}\right) + O(N^{-1})\right), \tag{3.101}$$

where $k = 0, 1, 2, \ldots$. Here F is a rescaling on the complementary error function defined in (2.35). The derivation of the above formulae is based on Stirling's formula and the following well known integral representation of the exponential polynomials in terms of the incomplete Gamma-function:

$$e_n(x) = e^x \frac{\Gamma(n+1, x)}{\Gamma(n+1)} = \frac{e^x}{n!}\int_x^\infty t^n e^{-t} dt, \; n = 0, 1, 2 \ldots, \tag{3.102}$$

see [29, Chapter 8.11.10] for more details. The calculations leading to (3.100) and (3.101) are straightforward, but lengthy due to the fact that we need to know the asymptotic expansion of e_{N+k} and $\log \frac{(N+\sqrt{N}a+b)^{N+1}}{(N+1)!}$ up to and including the terms of order $N^{-1/2}$, see also [29].

As a consequence of (3.101),

$$f_{N+1}(N+\sqrt{N}a+b) = \sqrt{\frac{N}{2\pi}}e^{N+\sqrt{N}a+b-\frac{a^2}{2}} \times \left(1-a\sqrt{2\pi}e^{\frac{a^2}{2}}F(a)\right)(1+O(N^{-\frac{1}{2}})), a,b\in\mathbb{C}. \tag{3.103}$$

Now we are ready to calculate the edge scaling limit of conditional overlaps. Let

$$\begin{aligned} x^{(N)} &= e^{i\theta}(\sqrt{N}+x), \\ y^{(N)} &= e^{i\theta}(\sqrt{N}+y), \\ \lambda^{(N)} &= e^{i\theta}(\sqrt{N}+\lambda), \end{aligned} \tag{3.104}$$

where $x,y,\lambda\in\mathbb{C}$ are of order unity. Then the edge scaling limit of the factor multiplying the determinant in the r.h.s. of (2.18) down-scaled by $N^{-1/2}$ is

$$\lim_{N\to\infty}\frac{e^{-|\lambda^{(N)}|^2}}{\pi\sqrt{N}}f_{N-1}(|\lambda^{(N)}|^2)=\frac{1}{\sqrt{2\pi^3}}\left(e^{-\frac{1}{2}(\lambda+\bar{\lambda})^2}-\sqrt{2\pi}\left(\lambda+\bar{\lambda}\right)F(\lambda+\bar{\lambda})\right),\ \lambda,\bar{\lambda}\in\mathbb{C}^2. \tag{3.105}$$

The derivation of (3.105) is based on (3.101). Note that (3.105) is valid for any pair of complex numbers $(\lambda,\bar{\lambda})$, not just on the real surface $\lambda=\bar{\bar{\lambda}}$, which makes it suitable for the calculation of both the diagonal and the off-diagonal overlaps.

To find the edge scaling limit of the kernel $K_{11}^{(N)}$ we substitute the expressions (3.104) into the formula for the kernel

$$K_{11}^{(N)}(x^{(N)},\bar{x}^{(N)},y^{(N)},\bar{y}^{(N)}\mid\lambda^{(N)},\bar{\lambda}^{(N)}) = \frac{1+(x^{(N)}-\lambda^{(N)})(\bar{x}^{(N)}-\bar{\lambda}^{(N)})}{\pi}e^{-x^{(N)}\bar{x}^{(N)}} \times G^{(N-1)}\left(\lambda^{(N)}\bar{\lambda}^{(N)},\frac{\bar{x}^{(N)}}{\bar{\lambda}^{(N)}},\frac{y^{(N)}}{\lambda^{(N)}}\right),$$

where $G^{(N)}$ is given by formula (3.87), and compute the large-N asymptotics using (3.100), (3.101) and (3.103). The result follows from another lengthy computation and is

$$K_{11}^{(N)}(x^{(N)},\bar{x}^{(N)},y^{(N)},\bar{y}^{(N)}\mid\lambda^{(N)},\bar{\lambda}^{(N)})=e^{\sqrt{N}(y-x)}\frac{1+(x-\lambda)(\bar{x}-\bar{\lambda})}{\pi}e^{-x\bar{x}+\bar{x}y} \times\frac{H\left(\lambda+\bar{\lambda},\lambda+\bar{x},\bar{\lambda}+y,y+\bar{x},(\lambda-y)(\bar{\lambda}-\bar{x})\right)}{(\lambda-y)^2(\bar{\lambda}-\bar{x})^2}\left(1+O(N^{-1/2})\right), \tag{3.106}$$

where the function H is defined in (2.36). We see that there the large-N limit of $K_{11}^{(N)}$ at the edge does not exist, but fortunately, the residual N-dependence can be eliminated by the N-dependent conjugation

$$K(x,\bar{x},y,\bar{y}\mid\lambda,\bar{\lambda})\to e^{\sqrt{N}x}K(x,\bar{x},y,\bar{y}\mid\lambda,\bar{\lambda})e^{-\sqrt{N}y}, \tag{3.107}$$

which does not change the value of the conditional overlap. Therefore we can conclude that

$$\begin{aligned} &K_{11}^{(edge)}(x,\bar{x},y,\bar{y}\mid\lambda,\bar{\lambda}):=\lim_{N\to\infty}e^{\sqrt{N}x}K_{11}^{(N)}(x^{(N)},\bar{x}^{(N)},y^{(N)},\bar{y}^{(N)}\mid\lambda^{(N)},\bar{\lambda}^{(N)})e^{-\sqrt{N}y} \\ &=\frac{1+(x-\lambda)(\bar{x}-\bar{\lambda})}{\pi}e^{-x\bar{x}+\bar{x}y}\frac{H\left(\lambda+\bar{\lambda},\lambda+\bar{x},\bar{\lambda}+y,y+\bar{x},(\lambda-y)(\bar{\lambda}-\bar{x})\right)}{(\lambda-y)^2(\bar{\lambda}-\bar{x})^2}. \end{aligned} \tag{3.108}$$

Substituting (3.105) and (3.108) into the edge scaling limit (2.26) of the conditional overlap $D_{11}^{(N,k)}$ we arrive at the statement (2.37), (2.38), (2.39) and (2.40) of the Corollary 2.

To find the edge scaling limit of the off-diagonal overlap $D_{12}^{(N,k)}$, we need to substitute its expression (2.22) into (2.27) and calculate the large-N limit of the resulting sequence. As before, the calculation reduces to the evaluation of the scaling limits of the pre-factor in the r.h.s. of (2.22) and the kernel (2.23).

A straightforward computation based on (3.103, 3.108) gives

$$\begin{aligned}
&\lim_{N\to\infty}(-1)\frac{e^{-|\lambda_1^{(N)}|^2-|\lambda_2^{(N)}|^2}}{\pi^2\sqrt{N}}f_{N-1}(\lambda_1^{(N)}\bar{\lambda}_2^{(N)})\kappa^{(N-1)}(\bar{\lambda}_1^{(N)},\lambda_2^{(N)}\mid\lambda_1^{(N)},\bar{\lambda}_2^{(N)})\\
&= -\frac{e^{-|\lambda_{12}|^2-\frac{1}{2}(\lambda_1+\bar{\lambda}_2)^2}}{\sqrt{2\pi^5}\lambda_{12}^2\bar{\lambda}_{12}^2}\left(1-\sqrt{2\pi}(\lambda_1+\bar{\lambda}_2)e^{\frac{1}{2}(\lambda_1+\bar{\lambda}_2)^2}F(\lambda_1+\bar{\lambda}_2)\right)\\
&\quad\times H(\lambda_1+\bar{\lambda}_2,\lambda_1+\bar{\lambda}_1,\lambda_2+\bar{\lambda}_2,\lambda_2+\bar{\lambda}_1,-\lambda_{12}\bar{\lambda}_{12}).
\end{aligned}\tag{3.109}$$

Similarly, introducing in addition to (3.104), $u^{(N)}=e^{\theta}(\sqrt{N}+u)$, $v^{(N)}=e^{\theta}(\sqrt{N}+v)$, we find that

$$\begin{aligned}
&K_{12}^{(N)}(x^{(N)},\bar{x}^{(N)},y^{(N)},\bar{y}^{(N)}\mid u^{(N)},\bar{u}^{(N)},v^{(N)},\bar{v}^{(N)})\\
&= e^{\sqrt{N}(y-x)}K_{12}^{(edge)}(x,\bar{x},y,\bar{y}\mid u,\bar{u},v,\bar{v})\left(1+O(N^{-1/2})\right),
\end{aligned}\tag{3.110}$$

where $K_{12}^{(edge)}$ is given by (2.42). Substituting (3.109) and (3.110) into the right hand side of (2.27), performing the N-dependent conjugation (3.107) of the kernel inside the resulting determinant and computing the $N\to\infty$ limit, we arrive at the statements (2.41), (2.42) of the Corollary 2. Corollary 2 is proved.

3.7 The proof of corollary 3

It follows from (2.28), (2.32), (2.37) and (2.41) that

$$D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = D_{11}^{(bulk,\ 1)}(\lambda_1)\det_{2\le i,j\le k}(K_{11}^{(bulk)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j\mid\lambda_1,\bar{\lambda}_1)),\ k\ge 1,\tag{3.111}$$

$$D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = D_{12}^{(bulk,\ 2)}(\lambda_1,\lambda_2)\det_{3\le i,j\le k}(K_{12}^{(bulk)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j\mid\lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2)),\quad k\ge 2,\tag{3.112}$$

$$D_{11}^{(edge,\ k)}(\boldsymbol{\lambda}^{(k)}) = D_{11}^{(edge,\ 1)}(\lambda_1)\det_{2\le i,j\le k}(K_{11}^{(edge)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j\mid\lambda_1,\bar{\lambda}_1)),\ k\ge 1,\tag{3.113}$$

$$D_{12}^{(edge,\ k)}(\boldsymbol{\lambda}^{(k)}) = D_{12}^{(edge,\ 2)}(\lambda_1,\lambda_2)\det_{3\le i,j\le k}(K_{12}^{(edge)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j\mid\lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2)),\quad k\ge 2,\tag{3.114}$$

where the expressions for all the relevant kernels are given in Corollaries 1 and 2. We see that the ratios

$$\frac{D_{11}^{(edge,\ k)}}{D_{11}^{(edge,\ 1)}},\ \frac{D_{12}^{(edge,\ k)}}{D_{12}^{(edge,\ 2)}},\ \frac{D_{11}^{(bulk,\ k)}}{D_{11}^{(bulk,\ 1)}}\ \text{and}\ \frac{D_{12}^{(bulk,\ k)}}{D_{12}^{(bulk,\ 2)}}\tag{3.115}$$

are completely determined by the conjugacy classes of the corresponding kernels. Therefore, the claim of the Corollary 3 is an immediate consequence of the following relations between the kernels:

$$\lim_{R\to-\infty} e^{(R+\bar\lambda)x} K_{11}^{(edge)}(R+x, R+\bar x, R+y, R+\bar y \mid R+\lambda_1, R+\bar\lambda_1) e^{-(R+\bar\lambda)y} = K_{11}^{(bulk)}(x,\bar x, y, \bar y \mid \lambda_1, \bar\lambda_1), \tag{3.116}$$

$$\lim_{R\to-\infty} K_{12}^{(edge)}(R+x, R+\bar x, R+y, R+\bar y \mid R+\lambda_1, R+\bar\lambda_1, R+\lambda_2, R+\bar\lambda_2) = K_{12}^{(bulk)}(x,\bar x, y,\bar y \mid \lambda_1,\bar\lambda_1,\lambda_2,\bar\lambda_2), \tag{3.117}$$

Both (3.116) and (3.117) can be derived from the following asymptotic formula for the H-function (2.36):

$$H(-\epsilon^{-1}+a, -\epsilon^{-1}+b, -\epsilon^{-1}+c, -\epsilon^{-1}+d, f) = \left(e^{-f}-1+f\right)(1+O(\epsilon)), \tag{3.118}$$

where $\epsilon>0, (a,b,c,d,f)\in\mathbb{C}^5$. This formula follows directly from the standard asymptotic for the complementary error function (2.35): for $a<0$,

$$F(a) = 1+O\left(e^{-a^2/2}\right),\ F'(a) = O\left(e^{-a^2/2}\right). \tag{3.119}$$

The proof of (3.116):

$$\begin{aligned}
&\lim_{R\to-\infty} e^{(R+\bar\lambda)x} K_{11}^{(edge)}(R+x, R+\bar x, R+y, R+\bar y \mid R+\lambda, R+\bar\lambda) e^{-(R+\bar\lambda)y} \\
= & -\lim_{R\to-\infty} e^{\bar\lambda x}\left(\frac{1+(x-\lambda)(\bar x-\bar\lambda)}{\pi z^2} e^{\bar x(y-x)} H\left(2R+a, 2R+b, 2R+c, 2R+d, z\right)\big|_{z=(\bar x-\bar\lambda)(y-\lambda)}\right) e^{-\bar\lambda y} \\
= & \frac{1+(x-\lambda)(\bar x-\bar\lambda)}{\pi} e^{-|x-\lambda|^2} \frac{1-(1-z)e^z}{z^2}\Bigg|_{z=(\bar x-\bar\lambda)(y-\lambda)} = K_{11}^{(bulk)}(x,\bar x,y,\bar y\mid\lambda,\bar\lambda).
\end{aligned}$$

To prove (3.117), let us first notice that (3.118) leads to

$$\kappa^{(edge)}(R+\bar x, R+y \mid R+\lambda, R+\bar\lambda) = e^{R^2+R(\bar x+y)-|\lambda|^2+\bar\lambda y+\lambda\bar x}\kappa^{(bulk)}(\bar x, y\mid\lambda,\bar\lambda)\left(1+O(R^{-1})\right), \tag{3.120}$$

where $R<0$. Also,

$$\begin{aligned}
\omega^{(edge)}(R+x, R+\bar x \mid R+\lambda, R+\bar\lambda) &= \frac{1+(x-\lambda)(y-\bar\lambda)}{\pi} e^{-|R+x|^2} \\
&= e^{-R^2-R(x+\bar x)+\lambda\bar\lambda-\bar x\lambda-x\bar\lambda}\omega^{(bulk)}(x,\bar x\mid\lambda,\bar\lambda).
\end{aligned} \tag{3.121}$$

Notice that the last two relations are still valid if $\kappa^{(edge)}$ and $\omega^{(edge)}$ are treated as functions on $\mathbb{C}^4$. Substituting (3.120), (3.121) into (2.42) we find

$$\begin{aligned}
&K_{12}^{(edge)}(R+x, R+\bar x, R+y, R+\bar y \mid R+\lambda_1, R+\bar\lambda_1, R+\lambda_2, R+\bar\lambda_2) \\
&\quad = e^{(R+\bar\lambda_2)(y-x)} K_{12}^{(bulk)}(x,\bar x,y,\bar y\mid\lambda_1,\bar\lambda_1,\lambda_2,\bar\lambda_2)\left(1+O(R^{-1})\right).
\end{aligned} \tag{3.122}$$

Therefore

$$\lim_{R\to-\infty} e^{(R+\bar\lambda_2)x} K_{12}^{(edge)}(R+x, R+\bar x, R+y, R+\bar y \mid R+\lambda_1, R+\bar\lambda_1, R+\lambda_2, R+\bar\lambda_2) e^{-(R+\bar\lambda_2)y} = K_{12}^{(bulk)}(x,\bar x,y,\bar y\mid\lambda_1,\bar\lambda_1,\lambda_2,\bar\lambda_2).$$

Equation (3.117) is established and therefore the Corollary 3 is proved.

3.8 The proof of relations (2.49), (2.50) and Corollary 4

We shall start with deriving (2.49), (2.50). According to Corollary 1,

$$D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = \frac{1}{\pi^k}\prod_{i=2}^{k}\left(\frac{1+|\lambda_{i1}|^2}{|\lambda_{i1}|^4}e^{-|\lambda_{i1}|^2}\right)\det_{2\le m,n\le k}\left[1-(1-\bar{\lambda}_{m1}\lambda_{n1})e^{\bar{\lambda}_{m1}\lambda_{n1}}\right]. \quad (3.123)$$

Let $\mathbf{1}$ be a $(k-1)$-dimensional column vector with all components equal to 1. Let M, A be $(k-1)\times(k-1)$ matrix. Let α be a constant. The next two identities follow directly from the block determinant formula (3.13):

$$\det(M-\mathbf{1}\otimes\mathbf{1}) = \det\left(\begin{array}{c|c} 1 & \mathbf{1}^T \\ \hline \mathbf{1} & M\end{array}\right), \quad (3.124)$$

$$\det\left(\begin{array}{c|c} 1 & \mathbf{1}^T \\ \hline \mathbf{1} & \alpha A\end{array}\right) = \alpha^{k-2}\det\left(\begin{array}{c|c} \alpha & \mathbf{1}^T \\ \hline \mathbf{1} & A\end{array}\right). \quad (3.125)$$

Applying the identities to the determinant in the r.h.s. of (3.123), one finds

$$\begin{aligned} D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) &= -\left(-\frac{1}{\pi}\right)^k\prod_{i=2}^{k}\left(\frac{1+|\lambda_{i1}|^2}{|\lambda_{i1}|^4}e^{-|\lambda_{i1}|^2}\right)\det_{2\le m,n\le k}\left(\begin{array}{c|c} 1 & \mathbf{1}^T \\ \hline \mathbf{1} & (1-\bar{\lambda}_{m1}\lambda_{n1})e^{\bar{\lambda}_{m1}\lambda_{n1}}\end{array}\right) \\ &= -\left(-\frac{1}{\pi}\right)^k\prod_{i=2}^{k}\left(\frac{1+|\lambda_{i1}|^2}{|\lambda_{i1}|^4}e^{-|\lambda_{i1}|^2+|\lambda_1|^2}\right)e^{-|\lambda_1|^2} \\ &\times\det_{2\le m,n\le k}\left(\begin{array}{c|c} e^{|\lambda_1|^2} & \mathbf{1}^T \\ \hline \mathbf{1} & (1-\bar{\lambda}_{m1}\lambda_{n1})e^{\bar{\lambda}_{m1}\lambda_{n1}-|\lambda_1|^2}\end{array}\right). \end{aligned} \quad (3.126)$$

Let

$$\mathfrak{D}_m := 1-\lambda_{m1}\frac{\partial}{\partial\lambda_m}, m=1,\dots k, \quad (3.127)$$

be a first order differential operator. Clearly,

$$[\mathfrak{D}_m,\mathfrak{D}_n]=0,$$

for $m,n\ge 1$. Observe also that

$$\mathfrak{D}_p e^{\bar{\lambda}_{m1}\lambda_{n1}} = \left(1-\delta_{pn}\bar{\lambda}_{m1}\lambda_{n1}\right)e^{\bar{\lambda}_{m1}\lambda_{n1}},\ \ p,m,n\ge 1.$$

These observations lead to the following identity:

$$\begin{aligned} \det\left(\begin{array}{c|c} e^{|\lambda_1|^2} & \mathbf{1}^T \\ \hline \mathbf{1} & (1-\bar{\lambda}_{m1}\lambda_{n1})e^{\bar{\lambda}_{m1}\lambda_{n1}-|\lambda_1|^2}\end{array}\right) &= \prod_{m=2}^{k}\mathfrak{D}_m\det\left(\begin{array}{c|c} e^{|\lambda_1|^2} & \mathbf{1}^T \\ \hline \mathbf{1} & e^{\bar{\lambda}_{m1}\lambda_{n1}-|\lambda_1|^2}\end{array}\right) \\ &= \prod_{m=2}^{k}\left(\mathfrak{D}_m e^{-\bar{\lambda}_m\lambda_1-\bar{\lambda}_1\lambda_m}\right)\det_{1\le p,q\le k}\left(e^{\bar{\lambda}_p\lambda_q}\right). \end{aligned} \quad (3.128)$$

Substituting (3.128) into (3.126) and simplifying we find:

$$D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = (-1)^{k-1}\prod_{i=2}^{k}\left(\frac{1+|\lambda_{i1}|^2}{|\lambda_{i1}|^4}\right)\prod_{n=2}^{k}e^{-|\lambda_{n1}|^2}\mathfrak{D}_{\mathfrak{n}}e^{|\lambda_{n1}|^2}\rho^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}), \quad (3.129)$$

where $\rho^{(bulk,\ k)}$ is the bulk scaling limit of the k-point correlation function (2.7),

$$\rho^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = \prod_{m=1}^{k} \frac{e^{-|\lambda_m|^2}}{\pi} \det_{1\le i,j\le k}\left(e^{\bar{\lambda}_i\lambda_j}\right), \tag{3.130}$$

see Ginibre's classical paper [19]. Notice that

$$e^{-|\lambda_{n1}|^2}\mathfrak{D}_{\mathfrak{n}} e^{|\lambda_{n1}|^2} = \mathfrak{D}_n - |\lambda_{n1}|^2.$$

Substituting this formula into (3.129) we arrive at the representation (2.49).

The easiest way to prove (2.50) is to recall that the bulk scaling limits $D_{11}^{(bulk,\ k)}$ and $D_{12}^{(bulk,\ k)}$ are still related via the formula (2.14) of Lemma 1. Applying the relation to both sides of (2.49) and noticing that

$$\hat{T}\rho^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = -e^{|\lambda_{12}|^2}\rho^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}), \tag{3.131}$$

we obtain formula (2.50).

Now we are ready to prove Corollary 4. The decay of correlations for the complex Ginibre ensemble is Gaussian,

$$\rho^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = \pi^{-k} + O(e^{-|\lambda_{ij}|^2}, 1 \le i < j \le k). \tag{3.132}$$

Therefore, for well separated eigenvalues,

$$\frac{\partial}{\partial\lambda_m}\rho^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = O(e^{-L^2}),\ 1 \le m \le k, \tag{3.133}$$

where $L = \inf_{1\le i\ne j\le k}|\lambda_{ij}|$. Substituting (3.132) and (3.133) into (2.49), (2.50) we immediately arrive at the exact algebraic asymptotic of conditional overlaps stated in Corollary 4.

4 Summary and Open Problems

We have analysed the overlap between left and right eigenvectors in the complex Ginibre ensemble of random matrices, conditioned on k complex eigenvalues. Starting from the results of Chalker and Mehlig we used a combination of the inversion of the moment matrix and theory of orthogonal polynomials in the complex plane, to arrive at a determinantal structure for the diagonal overlap. It is valid for finite matrix size N and fixed k and is explicitly given in terms of a kernel, containing combinations of exponential polynomials. Its analyticity lead us to deduce the off-diagonal overlap as a $k \times k$ determinant and its kernel as well.

These findings allowed us to take the microscopic limit both in the bulk and the edge of the spectrum. Both bulk and edge kernel were explicitly derived and conjectured to be universal. For the bulk we restricted ourselves to the vicinity of the origin, but due to the translational invariance of the limiting kernel we expect to find the same answer everywhere in the bulk of the spectrum. At the edge we found a residual rotational symmetry of the kernel, that is independent of the angle where at the circular edge at $|z| = \sqrt{N}$ we take the limit. At large argument separation the bulk limit answers also allowed us to derive the algebraic decay of the conditional overlaps and establish their asymptotic factorisation.

It is an open question if the determinantal structures that we found can also be obtained in more general ensembles at finite-N, such as is products of Ginibre matrices, see [10] for some conjectures (as well as [4] for large-N), or in a more general non-Gaussian setting. This will be a formidable task because, for example, for products of random matrices more and more off-diagonal elements of the moment matrix emerge.

A further question is concerning other symmetry classes. Whilst the most difficult real Ginibre ensemble has been addressed in [7, 17], for the quaternionic Ginibre ensemble so far only first steps have been taken [14, 3]. The structures found by Chalker and Mehlig persist, and the same result in the global, macroscopic regime is found for the diagonal and off-diagonal overlap. Note however, that already in the symmetry class of complex matrices these are non-universal when going to non-Gaussian ensembles, cf. [4]. It remains to be seen if a Pfaffian structure similar to the one in the present work can be found. In principle, the building block, a Pfaffian formula for expectation values of products of characteristic polynomials prevails [2], that generalises the Christoffel type theorem [1] from orthogonal to skew orthogonal polynomials in the complex plane.

Our motivation was, apart from finding integrable structures, the coupled stochastic motion of the complex eigenvalues and corresponding eigenvectors. Its further investigation is left for future work.

Acknowledgements. We would like to thank Yan Fyodorov and Mario Kieburg for fruitful discussions at an early stage of this project. Furthermore, partial support from the following grants is gratefully acknowledged: CRC1283 "Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications" by the German Research Foundation DFG for Gernot Akemann, Roger Tribe is partially supported by a Leverhulme Research Fellowship RF-2-16-655.

A Appendix: Normal Evolutions

The fact that Brownian motion with values in normal matrices is related to the 2-dimensional log-gas seems to be a common albeit unpublished knowledge. Here we present the verification of this fact just to emphasise the difference between the evolution equations and the 'Dysonian dynamics' for the complex Ginibre ensemble studied in [20].

Let $M = \{H \in \mathbb{C}^{N^2} \mid [H, H^\dagger] = 0\}$ be the space of $N \times N$ normal matrices. Notice that the space M is not linear. It is not a smooth manifold either - there are singularities at the points of degeneration of the spectrum of H. It must still be possible to construct an M-valued Brownian motion, provided that it does not visit the singular points. We can attempt to define this process by using the Laplace-Beltrami operator (which exists on the complement to the set of singular points) as the generator. Let M_c be the complement to the set of singularities of M. For that we need to choose good local coordinates on M. A reasonable choice would be

$$H = U\Lambda U^\dagger, \tag{A.1}$$

where Λ is a complex diagonal matrix with entries $\lambda^1, \lambda^2, \ldots, \lambda^N$ and $U \in U(N)/U(1)^N$. The Riemannian metric on M_c is induced by the embedding $M_c \subset \mathbb{C}^{N^2}$,

$$G(\delta H, \delta H) = \mathrm{Tr}\delta H \delta H^\dagger. \tag{A.2}$$

But for $H \in M$,

$$\delta H = U(\delta\Lambda + [\delta g, \Lambda])U^\dagger, \tag{A.3}$$

where $\delta g = U^\dagger \delta U \in Lie(U(N))$, with the additional restriction $\delta g^{ii} = 0$, $i = 1, 2, \ldots, N$. Therefore,

$$G(\delta H, \delta H) = \sum_i |\delta\Lambda_{ii}|^2 + 2\sum_{i<j} |\lambda^i - \lambda^j|^2 |\delta g^{ij}|^2. \tag{A.4}$$

(In this Section only we use superscripts to label λ's in line with the labelling convention for local coordinates in differential geometry.) We see that the metric tensor is diagonal in (Λ, U)-coordinates and that

$$\sqrt{\det(G)} = 2^{\frac{N(N-1)}{2}} |\Delta^{(N)}(\Lambda)|^2. \tag{A.5}$$

The inversion of G is straightforward and the generator of the Brownian motion on M_c is given by

$$L = \frac{1}{|\Delta^{(N)}(\Lambda)|^2}\sum_{i=1}^{N}\left(\frac{\partial}{\partial\lambda^i}|\Delta^{(N)}(\Lambda)|^2\frac{\partial}{\partial\overline{\lambda}^i}+\frac{\partial}{\partial\overline{\lambda}^i}|\Delta^{(N)}(\Lambda)|^2\frac{\partial}{\partial\lambda^i}+\frac{1}{2}\sum_{i<j}\frac{1}{|\lambda^i-\lambda^j|^2}\frac{\partial}{\partial g^{ij}}\frac{\partial}{\partial\bar{g}^{ij}}\right). \tag{A.6}$$

There are two obvious points to notice: (i) The dynamics of eigenvalues is Markovian; (ii) Unitary degrees of freedom speed up near the singular points, i.e. at $\lambda^i = \lambda^j$ for some $i \neq j$.

The generator for the eigenvalue dynamics is

$$L = \frac{1}{\Delta^{(N)}(\Lambda)}\sum_{i=1}^{N}\frac{\partial}{\partial\lambda^i}\frac{\partial}{\partial\overline{\lambda}^i}\Delta^{(N)}(\Lambda)+\frac{1}{\Delta^{(N)}(\overline{\Lambda})}\sum_{i=1}^{N}\frac{\partial}{\partial\lambda^i}\frac{\partial}{\partial\overline{\lambda}^i}\Delta^{(N)}(\bar{\Lambda}). \tag{A.7}$$

The corresponding system of SDE's is

$$d\lambda^i = 2\sum_{k\neq i}\frac{1}{\overline{\lambda}^i-\overline{\lambda}^k}dt+\sqrt{2}dW_t^i, \tag{A.8}$$

$$d\bar{\lambda}^i = 2\sum_{k\neq i}\frac{1}{\lambda^i-\lambda^k}dt+\sqrt{2}d\overline{W}_t^i, \tag{A.9}$$

where $\{W_t^i, \overline{W}_t^i\}_{t\geq 0}^{1\leq i\leq N}$ are independent complex Brownian motions with non-zero covariances $\mathbb{E}\left(\bar{W}_t^i W_s^j\right) = \delta^{ij} s\wedge t$. Equations (A.8) and (A.9) are valid until the time of the first exit from M_c, which is likely to be infinite, but we do not check it here.

It is well known [12], that the $t = 1$ marginal distribution of eigenvalues for normal evolutions (the so called normal random matrix model) coincides with the law (1.4) for the complex Ginibre eigenvalues. (This correspondence breaks down for models with non-Gaussian potentials.) Yet the evolution equations (A.8, A.9) are very obviously different from the equations for the joint evolution of eigenvalues and eigenvectors for the complex Ginibre evolutions derived in [20].

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